ON EPIMORPHISMS AND DOMINIONS OF SEMIGROUPS

ABSTRACT
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SUBMITTED FOR THE AWARD OF THE DEGREE OF

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BY

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Under the Supervision of

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<u>*ABSTRACT*</u>

The aim of this thesis is mainly to investigate the following three questions:

(1) *Which classes of semigroups are closed or in other words which classes of semigroups have the special amalgamation property?*

(2) *What type of semigroup classes are saturated and supersaturated?*

(3) *Which classes of semigroups are absolutely closed ?*

However, a short proof of celebrated Zigzag Theorem for the category of all commutative semigroups is also provided.

The present exposition consists of six chapters and each chapter is divided into various sections.

Most of the new material presented in Chapters 2, 3, 4, 5 and 6 of this thesis, except where reference has been given to other sources, appears in the following papers.

Epimorphisms, closed and supersaturated semigroups, Submitted:
Chapter 2, Sections 2.2 and 2.4.
Chapter 4, Section 4.3.

Special Semigroup Amalgams of Quasi Unitary Subsemigroups and of Quasi Normal Bands, Asian European Journal of Mathematics, under revision:
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Chapter 3, Sections 3.2 and 3.3 and Section 3.4.

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Chapter 2, Section 2.4; Chapter 5, Section 5.3 and Chapter 6, Section 6.3

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Epimorphisms, dominions and classes of closed regular semigroups, Submitted: Chapter 6, Section 6.2.

Conversely, all the new material in the above papers appears in the thesis.

Chapter 1 contains introductory concepts of semigroup theory and some important results, including a full proof of Isbell's Zigzag Theorem on which the whole of the thesis is based.

In Chapter 2, we partially answer question (1) by investigating certain classes of semigroups which are closed.

In [45], Howie showed that the amalgam $[S, T; U]$ is embeddable if U is almost unitary in S and $V = (U\phi)$ is almost unitary in T which implied that almost unitary subsemigroup of a semigroup is closed in the containing semigroup. In Section 2.2, we provide, by using zigzag manipulations, new and direct proof of this result and show that the special semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is embeddable if U is almost unitary in S. In Section 2.3, we provide, by zigzag manipulations, a new and direct algebraic proof of [79, Corollary 6.5] and show that the special semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is embeddable if U is quasi-unitary in S and, thus, generalize the result of Section 2.2. In Section 2.4, we have studied some closed classes of bands. Scheiblich [81] had shown that the class of all normal bands is closed. We extend this result to left[right] regular bands, left[right] quasinormal bands and $WL[WR]$ regular bands. We also show that the class of all left[right] regular bands is closed in the class of all regular bands. Then, we generalize this result by establishing that the class of all left[right] regular bands is closed in the class of all left[right] semiregular bands. In Sections 2.5 and 2.6, we find some closed classes of semigroups that satisfy a heterotypical identity and a homotypical identity respectively.

In Chapter 3, Section 3.2, we determine that the class of all inflations of Clifford semigroups is saturated while in the next section, we show that a subclass of the class of all semigroups satisfying the identity $ax = axa[xa = axa]$ is saturated. Finally, in Section 3.4, we show that the class of all quasi-commutative semigroups satisfying a nontrivial identity of which at least one side has no repeated variable is saturated.

In Chapter 4, we discuss ideals and supersaturated semigroups. In Section 4.2, we first present an example due to Higgins [38], of a supersaturated semigroup, and, then, give a brief exposition of semigroup amalgams and their relationship with dominions. In [37], Higgins showed that a semigroup U is saturated [supersaturated] if the ideal U^n is saturated [supersaturated]. Whether or not the converse holds, is an open problem. In [38], Higgins showed that the converse holds in some cases and proved that if S is a supersaturated semigroup, then any commutative globally idempotent ideal of S is also supersaturated.

Khan and Shah [66] generalized this result from commutative ideals to permutative ideals by taking U as a permutative globally idempotent ideal satisfying a permutation identity $x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}$ for which $i_1 = 1$ and $i_n \neq n$. In Section 4.3, we extend this result by taking U as a permutative globally idempotent ideal satisfying a seminormal permutation identity and, thus, relax the right semicommutativity of U. We further extend this result and enlarge the class of supersaturated globally idempotent ideals of a supersaturated semigroup by showing that a globally idempotent ideal of a supersaturated semigroup satisfying the identity $axa = ax[axa = xa]$ is supersaturated.

In Chapter 5, Section 5.2, we give a new and short proof of Isbell's Zigzag Theorem for the category of all commutative semigroups while the next section deals with some results on absolutely closed semigroups.

In Chapter 6, we have found some classes of regular semigroups which have the special amalgamation property. In Section 6.2, we show that a regular subsemigroup of a semigroup satisfying some condition in the containing semigroup is closed in the containing semigroup. Then, we prove that the class of all left[right] Clifford semigroups, $\mathcal{R}[\mathcal{L}]$-unipotent semigroups and left [right] quasinormal orthodox semigroups have special amalgamation property. We also, prove that the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is strongly embeddable in the class of all left[right] semiregular orthodox semigroups, which imply that the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is strongly embeddable in the class of all quasi-inverse semigroups. Finally, we show that the classes of all WL[WR]-regular orthodox semigroups and that of all left[right] seminormal orthodox semigroups have special amalgamation property.

Section 6.3, includes some results based on closedness of the class of all bands within the class of all semigroups satisfying some homotypical identities. First, we show that the class of all normal bands is closed within the class of all medial semigroups, generalizing the long known fact that the class of all normal bands is closed (see [81]). Then, we show that the class of all left[right] seminormal bands is closed within the class of all semigroups satisfying the identity $axy = axyay[yxa = yayxa]$ which imply, as a corollary, that the class of all left[right] seminormal bands is closed.

At the end, an exhaustive reference of the literature consulted during this work, has been given.

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Certificate

*This is to certify that the thesis entitled "ON EPIMORPHISMS AND DOMINIONS OF SEMIGROUPS" is based on a part of research work done by **Mr. Noor Alam** and carried out under my guidance in the Department of Mathematics, Aligarh Muslim University, Aligarh. To the best of my knowledge, the work presented in the thesis is original and has not been submitted to any other university or institution for the award of a degree.*

*It is further certified that **Mr. Noor Alam** has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of the Aligarh Muslim University, Aligarh.*

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August, 2012

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(NOOR ALAM)

PREFACE

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At the end, an exhaustive reference of the literature consulted during this work, has been given.

CHAPTER 1

PRELIMINARIES

§ 1.1. INTRODUCTION

The group theory and ring theory have been adopted as a model by semigroup Theorists initially as its evidence is manifested by earliest major contributions to the semigroup theory made by (Suschkewitz [87], Rees [77], Clifford [15], and Dubreil [19]). However, in more recent years, the characteristics aims and methods developed in the subject are bearing vague witness to link with aforesaid parts of abstract algebra which is mainly due to the necessity of studying congruences. In a group, a congruence is determined if we know a single congruence class, in particular if we know the normal subgroup which is the class containing the identity. Similarly, in a ring, a congruence is determined if we know the ideal which is the congruence class containing the zero. In semigroups there is no such fortunate occurrence, and, therefore, we are faced with the necessity of studying congruence as such. That is why this necessity gives semigroup theory its characteristic flavour. As semigroups are first and simplest type of algebra, to which the methods of universal algebra must be applied, and any mathematician interested in universal algebra will find semigroup theory a rewarding study.

This chapter is devoted to collect some basic semigroup theoretic notions and results with a view to make our thesis as self contained as possible, whereas the elementary knowledge of the algebraic concepts such as groups, homomorphisms etc. has been preassumed, and thus, no attempt is being made to discuss them here. Most of the material included in this chapter occurs in the standard literature, namely Clifford and Preston [16], Howie ([44]-[54]), Hall ([22]-[29]), Higgins ([30]-[43]), Khan ([58]-[66]) and Petrich [71].

§ 1.2. BASIC DEFINITIONS

In this section, we give a brief exposition of some basic definitions and terminology of semigroup theory.

Definition 1.2.1. A *semigroup* $(S, \circ)$ is a non-empty set S together with an associative binary operation " $\circ$ ".

In accordance with the usual practice, we often speak simply of a semigroup S when the operation " $\circ$ " is understood and abbreviate the product $a \circ b \ (a,\ b \in S)$ as ab.

Definition 1.2.2. A semigroup S is said to be *commutative* if

$$xy = yx, \quad \forall\ x,\ y \in S.$$

Definition 1.2.3. A non-empty subset T of a semigroup S is called a *subsemigroup* of S if

$$xy \in T,\ \forall\ x,\ y \in T.$$

This condition can be expressed more compactly as $T^2 \subseteq T$. The associativity that holds throughout S certainly holds throughout T and so, T itself is a semigroup.

Definition 1.2.4. A subsemigroup of S which is a group with respect to the operation inherited from S is called a *subgroup* of S.

It is easy to see that a non-empty subset T of a semigroup S is a subgroup of S if and only if

$$(\forall\ a \in T),\ aT = T \text{ and } Ta = T.$$

Definition 1.2.5. If a semigroup S contains an element e such that

$$ex = xe = x, \quad (\forall\ x \in S),$$

we say that e is an *identity element* (or just an *identity*) of S, and S is said to be a *semigroup with identity* or (more usually) a *monoid.*

Like groups, an identity element of a semigroup, if it exists, is unique.

Definition 1.2.6. If a semigroup S has no identity element, then we can easily adjoin an extra element 1 to S to form a monoid, by defining

$$1s = s1 = s, \quad \forall\, s \in S, \quad \text{and} \quad 11 = 1.$$

Thus $S \cup \{1\}$ becomes a monoid. We now define

$$S^1 = \begin{cases} S & \text{if S has an identity element} \\ \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

We refer to S^1 as the monoid obtained from S by adjoining an identity, if necessary.

Definition 1.2.7. If a semigroup S with at least two elements contains an element 0 such that

$$0x = x0 = 0, \ \ (\forall\, x \in S),$$

then we say that 0 is a *zero element* (or just *zero*) of S and S is said to be a *semigroup with zero.*

A zero element of a semigroup, if exist, is also unique.

Definition 1.2.8. If a semigroup S has no zero element, then we can adjoin an extra element 0, and define

$$0s = s0 = 0, \ \forall\, s \in S, \ \text{and}\ 00 = 0.$$

It is a routine matter to check that $S \cup \{0\}$ is a semigroup with zero. By analogy with S^1, we define

$$S^0 = \begin{cases} S & \text{if S has a zero element} \\ \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

Again, we refer to S^0 as the semigroup obtained from S by adjoining a zero, if necessary.

Definition 1.2.9. An element e of a semigroup S is called an *idempotent* element if $e^2 = e$.

Definition 1.2.10. A semigroup S is said to be *globally idempotent* if for all $s \in S$, there exist $x, y \in S$ such that $s = xy$ or equivalently $S^2 = S$.

Definition 1.2.11. A semigroup S is said to be a *band* if every element of S is an idempotent.

Definition 1.2.12. A semigroup S is said to be a *left zero semigroup* if $ab = a$, $\forall$ $a,\ b \in S$. *Right zero semigroups* are defined dually.

Throughout the text, by a bracketed statement, we shall mean a statement dual to the other statement.

Definition 1.2.13. If I and Λ are non-empty sets, then we may define an associative binary operation " $\circ$ " on $I \times \Lambda$ as:

$$(i_1, \lambda_1) \circ (i_2, \lambda_2) = (i_1, \lambda_2), \quad \forall\, i_1,\ i_2 \in I;\ \lambda_1,\ \lambda_2 \in \Lambda.$$

Then $(I \times \Lambda,\ \circ)$ is a semigroup which is called a *rectangular band.*

If $\mid \Lambda \mid = 1$ $[\mid I \mid = 1]$, then the rectangular band $I \times \Lambda$ is a left [right] zero semigroup.

Definition 1.2.14. A band S is called a *semilattice* if $xy = yx$ $\forall\ x,\ y \in S$; a *left [right] normal band* if $abc = acb$ $[abc = bac]$ $\forall\ a,\ b,\ c \in S$ and a *normal band* if $axya = ayxa$ $\forall\ a,\ x,\ y \in S$.

Definition 1.2.15. A band S is called a *left [right] regular band* if

$$axa = ax \ \ [axa = xa] \ \ \forall\ a,\ x \in S.$$

Definition 1.2.16. Let S be a semigroup. An element a of S is said to be *regular* if there exists $x \in S$ such that $a = axa$. A semigroup whose all elements are regular is called a *regular semigroup.*

For example, the semigroup of all mappings of a non-empty set into itself, with respect to the operation of composition of maps, is a regular semigroup.

Definition 1.2.17. Let S be a semigroup. If a and b are elements of S, we say that b is an *inverse* of a if $aba = a, bab = b$. A semigroup S is said to be an *inverse semigroup* if each a in S has a unique inverse. We denote the unique inverse of a by a^{-1}.

Such a semigroup is certainly regular, but not every regular semigroup is an inverse semigroup. A rectangular band is an obvious example in which every element is an inverse of every other element.

Definition 1.2.18. A *Clifford semigroup* S is a regular semigroup whose idempotents lie in its center. i.e. $ex = xe$, $\forall\ e \in E(S)$ and $\forall\ x \in S$, where $E(S)$ is the set of all idempotents of S.

Definition 1.2.19. An *orthodox semigroup* is defined as a regular semigroup in which the idempotents form a subsemigroup.

The class of orthodox semigroups includes both the class of all inverse semigroups as well the class of all bands.

Result 1.2.20 ([51, Ch.6, Proposition 1.4]). If S is an orthodox semigroup, e is an idempotent and $a \in S$, then, for every inverse a' of a, elements $a'ea$ and aea' are both idempotents.

Definition 1.2.21. A *generalized inverse semigroup* is a regular semigroup whose idempotents form a normal band.

Definition 1.2.22. A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric and transitive.

Definition 1.2.23. Let S be a semigroup. A relation R on S is called *left compatible* (with the operation on S) if

$$(\forall\ s,\ t,\ a \in S)\quad (s,t) \in R \text{ implies } (as, at) \in R,$$

and *right compatible* if

$$(\forall\ s,\ t,\ a \in S)\quad (s,t) \in R \text{ implies } (sa, ta) \in R.$$

It is called *compatible* if

$$(\forall\ s,\ t,\ s',\ t' \in S)\ \ [(s,t) \in R \ \text{ and } \ (s',t') \in R] \ \text{ implies } \ (ss',tt') \in R.$$

A left [right] compatible equivalence relation on a semigroup S is called a *left [right] congruence* on S and a compatible equivalence relation on S is called a *congruence* on S.

Result 1.2.24 ([53, Proposition. 1.5.1]). An equivalence relation ρ on a semigroup S is a congruence if and only if it is both left and right compatible.

Definition 1.2.25. Let X be any set, and let F_X consists of all finite sequences of elements of X. If $(x_1,\ x_2,\ldots,x_m)$ and $(y_1,\ y_2,\ldots,y_n)$ be any two elements of F_X $(x_i, y_j \in X)$, where $(i = 1,\ 2,\ldots,m;\ j = 1,\ 2,\ldots,n)$, then we define their product by simple juxtaposition:

$$(x_1,\ x_2,\ldots,x_m)(y_1,\ y_2,\ldots,y_n) = (x_1,\ x_2,\ldots,x_m,\ y_1,\ y_2,\ldots,y_n).$$

F_X, thereby, becomes a semigroup which we call as the *free semigroup* on X. An element of F_X will be called as a *word* in the alphabet set X.

Definition 1.2.26. A *semigroup identity* $u = v$ is the formal equality of two words u and v formed by letters over an alphabet set X.

Definition 1.2.27. A semigroup S is said to *satisfy an identity* if for every substitution of elements from S for the letters forming the words of the identity, the resulting words are equal in S.

or, equivalently:

Definition 1.2.28. Let X be a countably infinite set and let F_X be the free semigroup on X. Let S be any semigroup. If $u,\ v \in F_X$, then we shall say that the identical relation (or identity) $u = v$ *is satisfied* in S if $u\phi = v\phi$ for every homomorphism $\phi : F_X \longrightarrow S$.

Definition 1.2.29. The class of semigroups, in which a finite or an infinite collection $u_1 = v_1,\ \ u_2 = v_2,\ldots$ of identical relations is satisfied, is called the *variety of* *semigroups determined by these identical relations*, and the list of identical relations is called a *presentation of the variety*, denoted by $[u_1 = v_1, \ \ u_2 = v_2, \ldots]$.

We shall take Birkhoff's Theorem for (semigroup) varieties for granted.

Result 1.2.30 ([21, Ch. 1 section 26, Theorem 3]). A non-empty class $\mathcal{V}$ of semigroups is a variety if and only if
(a) every subsemigroup of a semigroup in $\mathcal{V}$ is in $\mathcal{V}$;
(b) every homomorphic image of a semigroup in $\mathcal{V}$ is in $\mathcal{V}$;
(c) the direct product of a family of semigroups in $\mathcal{V}$ is in $\mathcal{V}$.

Definition 1.2.31. Let u be any word. The *content* of u is the (necessarily finite) set of all variables appearing in u, and will be denoted by $C(u)$. Further, for any variable x in u, $|x|_u$ will denote the number of occurrences of the variable x in the word u.

Definition 1.2.32. An identity

$$u(x_1, \ x_2, \ldots, x_n) = v(x_1, \ x_2, \ldots, x_n),$$

in the variables $x_1, x_2, \ldots, x_n$ is called *homotypical* if $C(u) = C(v)$ and *heterotypical* otherwise.

Definition 1.2.33. By a *permutation identity* in the variables $x_1, x_2, \ldots, x_n$ $(n \geq 2)$, we mean an identity

$$x_1 x_2 \cdots x_n = x_{i_1} x_{i_2} \cdots x_{i_n}, \tag{1.2.1}$$

where i is any permutation of the set $\{1, \ 2, \ldots, n\}$ and i_k, for any $1 \leq k \leq n$, denotes the image of k under the permutation i. Further, the identity (1.2.1) is said to be *non-trivial* if the permutation i is different from the identity permutation.

The following are some of the well known permutation identities;

$$\begin{array}{lcll}
x_1 x_2 & = & x_2 x_1 & \text{[commutativity];} \\
x_1 x_2 x_3 & = & x_1 x_3 x_2 & \text{[left normality];} \\
x_1 x_2 x_3 & = & x_2 x_1 x_3 & \text{[right normality];} \\
x_1 x_2 x_3 x_4 & = & x_1 x_3 x_2 x_4 & \text{[normality].}
\end{array}$$

Definition 1.2.34. A semigroup S is said to be a *permutative semigroup* if it satisfies a non-trivial permutation identity.

For further details and other related results on varieties and identities of semigroups, one may refer to [1], ([10] - [14]), [17], [74], [76], [83] and [92].

§ 1.3. EPIMORPHISMS AND DOMINIONS

Let U be a subsemigroup of a semigroup S. Following Isbell [57], we say that U *dominates* an element d of S if for every semigroup T and for all homomorphisms $\beta,\ \gamma : S \longrightarrow T,\ u\beta = u\gamma$ for each $u \in U$ implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the *dominion* of U in S and we denote it by $Dom(U, S)$. It may be easily seen that $Dom(U, S)$ is a subsemigroup of S containing U.

Following Howie and Isbell [54], a semigroup U is said to be *closed* in S if $Dom(U, S) = U$. Let $\mathcal{C}$ be a class of semigroups. A semigroup U is said to be $\mathcal{C}$-*closed* if for all $S \in \mathcal{C}$ such that U is a subsemigroup of S, $Dom(U, S) = U$. Let $\mathcal{B}$ and $\mathcal{C}$ be classes of semigroups such that $\mathcal{B} \subseteq \mathcal{C}$. Then $\mathcal{B}$ is said to be $\mathcal{C}$-*closed* if every member of $\mathcal{B}$ is $\mathcal{C}$-*closed*.

A semigroup U is said to be ***absolutely closed*** if it is closed in every containing semigroup S. At the other extreme, U is said to be *dense* or *epimorphically embedded* in S if $Dom(U, S) = S$. A semigroup U is said to be *saturated* if it cannot be properly epimorphically embedded in any properly containing semigroup S, that is, $Dom(U, S) \neq S$ for every properly containing semigroup S.

A morphism $\alpha : A \longrightarrow B$ in the category $\mathcal{C}$ of semigroups is said to be an *epimorphism* (*epi* for short) if for all morphisms $\beta,\ \gamma : B \longrightarrow C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$.

Remark 1.3.1. It is easy to see that onto morphisms are epimorphisms. Whether or not the converse is true, depends on the category under consideration. It is true in the categories of Sets, Abelian Groups and Groups for instance.

In general, epimorphisms are not onto in the categories of semigroups and rings.

Here epimorphisms can be characterized in term of so called "zigzags", a special sequence of factorizations of elements in the epimorphic image.

In the last section, we include a full proof of the Zigzag Theorem for semigroups, a result due to Isbell [57].

The following example of a semigroup epimorphism which is not onto appears in Drbohlav [18].

Example 1.3.2. Take the embedding i of the real interval $(0,\ 1]$ into $(0,\ \infty]$, where both are considered as multiplicative semigroups. To see that $i : (0,\ 1] \longrightarrow (0,\ \infty]$ is epi, take any pair of homomorphisms α, β from $(0,\ \infty]$ such that $i\alpha = i\beta$; that is, α and β agree on $(0,\ 1]$. We shall show that for any $x > 1$, $x\alpha = x\beta$. Let $x > 1$. Then

$$[(x)\alpha(1/x)\alpha](x)\beta = (1)\alpha(x)\beta = (1)\beta(x)\beta = (x)\beta.$$

Equally though, since $1/x < 1$,

$$[(x)\alpha(1/x)\alpha](x)\beta = (x)\alpha[(1/x)\alpha(x)\beta] = (x)\alpha[(1/x)\beta(x)\beta]$$

$$= (x)\alpha(1)\beta = (x)\alpha(1)\alpha = (x)\alpha.$$

Therefore, $\alpha = \beta$ and so i is epi.

Moreover, the embedding of an infinite monogenic semigroup into an infinite cyclic group, and the embedding (under multiplication) of the natural numbers into the positive rational numbers are other examples of this kind. However, Hall [28] has unified all these examples by showing that if U is a full subsemigroup (a subsemigroup that contains all the idempotent elements of the containing semigroup) of an inverse semigroup S, which generates S as an inverse semigroup, then the embedding of U in S is an epimorphism.

A class $\mathcal{C}$ of semigroups is said to be *epimorphically closed* if for all $S \in \mathcal{C}$ and $\alpha : S \longrightarrow T$ is epi implies $T \in \mathcal{C}$. Further, a class $\mathcal{C}$ of semigroups is called *saturated* if all of its members are saturated.

One may easily check that a morphism $\alpha : S \longrightarrow T$ is epi if and only if the inclusion $i : S\alpha \longrightarrow T$ is epi and the inclusion $i : U \longrightarrow S$ is epi if and only if $Dom(U, S) = S$.

It is clear that every absolutely closed class of semigroups is saturated and every saturated class is epimorphically closed, but the converse is not true in general. For example, the variety of all commutative semigroups is epimorphically closed ([57, Corollary 2.5]), but not saturated as the inclusion map of an infinite monogenic semigroup into an infinite cyclic group is epi [51, Ch.VII (Exercise 2(i))].

§ 1.4. SPECIAL SEMIGROUP AMALGAMS AND SOME IMPORTANT RESULTS

The study of epimorphisms and semigroup amalgams is not as isolated a topic as it may first appear, since it is equivalent to the study of so-called special amalgams. indeed, the study of dominions provides a doorway to the study of general semigroup amalgams, a broad and fundamental area of research.

To explain this fully, we introduce the appropriate definitions.

Definition 1.4.1. A *(semigroup) amalgam* $\mathcal{A} = [\{S_i : i \in I\};\ U;\ \{\phi_i : i \in I\}]$ consists of a semigroup U (called the *core* of the amalgam), a family $\{S_i : i \in I\}$ of semigroups disjoint from each other and from U, and a family $\phi_i : U \rightarrow S_i (i \in I)$ of monomorphisms. We shall simplify the notation to $\mathcal{U} = [S_i;\ U;\ \phi_i]$ or to $\mathcal{U} = [S_i;\ U]$ when the context allows.

We shall say that the amalgam $\mathcal{A}$ is *embedded* in a semigroup T if there exist a monomorphism $\lambda : U \rightarrow T$ and, for each $i \in I$, a monomorphism $\lambda_i : S_i \rightarrow T$ such that
(a) $\phi_i\lambda_i = \lambda$ for each $i \in I$;
(b) $S_i\lambda_i \cap S_j\lambda_j = U\lambda$ for all $i, j \in I$ such that $i \neq j$.

Thus, a semigroup amalgam can be thought of as an indexed family $\{S_i : i \in I\}$ of semigroups intersecting in a common subsemigroup U.

We say that the amalgam $[S_i;\ U]$ is *weakly embeddable* in a semigroup T if there are monomorphisms $\lambda_i : S_i \to T$, $\lambda_j : S_j \to T$ which agree on U; if further $S_i\lambda_i \cap S_j\lambda_j = U\lambda$ for all i, j, then we say that $[S_i;\ U]$ is *strongly embeddable* in T.

Definition 1.4.2. If every amalgam of semigroups from a class $\mathcal{C}$ of semigroups is weakly [strongly] embeddable in some semigroup $P \in \mathcal{C}$, then $\mathcal{C}$ is said to have the *weak [strong] amalgamation property.* A *weak [strong] amalgamation base*, for a class $\mathcal{C}$ of semigroups is a semigroup $U \in \mathcal{C}$ such that every amalgam of the form $[S, T;\ U]$ from $\mathcal{C}$ is weakly [strongly] embeddable in a semigroup $P \in \mathcal{C}$.

Definition 1.4.3. A semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ consisting of a semigroup S, a subsemigroup U of S, an isomorphic copy S' of S, where $\alpha : S \to S'$ be an isomorphism and i is the inclusion mapping of U into S, is called a *special semigroup amalgam.* A class $\mathcal{C}$ of semigroups is said to have the *special amalgamation property* if every special semigroup amalgam in $\mathcal{C}$ is strongly embeddable in a member of $\mathcal{C}$, and U is said to be a *special amalgamation base* in $\mathcal{C}$ if every special amalgam in $\mathcal{C}$ with core as U is strongly embeddable in $\mathcal{C}$.

It is well known that for the class $\mathcal{C}$ of all semigroups, each weak amalgamation base is also a strong amalgamation base (see [35]).

Remark 1.4.4. A special semigroup amalgam $[\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is always weakly embedded in S ([51, Chapter VII, Proposition 2.1]).

In the last of this section, we give some important results which are central for the work presented in the thesis. The following celebrated result due to Isbell, known as Isbell's Zigzag Theorem, is of basic importance to our investigations and is the main tool used for studying semigroup dominions. This theorem is so important for our purposes that we include a full proof of it, although this proof may be found in the introductory text of Howie [51].

Result 1.4.5 ([57, Theorem 2.3] or [51, Theorem VII. 2.13]). Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorization of d as follows:

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m}, \qquad (1.4.1)$$

where $m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m),\ \ y_i, t_i \in S\ (i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$

$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a *zigzag* in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

Proof. The proof in the reverse direction is just a straight forward zigzag manipulation. Suppose Z is a zigzag with value d in S over U and that $\alpha,\ \beta : S \longrightarrow T$ are two semigroup morphisms such that $\alpha|U = \beta|U$. Then

$d\alpha = (a_0t_1)\alpha = a_0\alpha t_1\alpha = a_0\beta t_1\alpha = (y_1a_1)\beta t_1\alpha = y_1\beta a_1\beta t_1\alpha = y_1\beta a_1\alpha t_1\alpha = y_1\beta(a_1t_1)\alpha = y_1\beta(a_2t_2)\alpha = \cdots = y_m\beta(a_{2m-1}t_m)\alpha = y_m\beta a_{2m}\alpha = y_m\beta a_{2m}\beta = (y_ma_{2m})\beta$ $=d\beta$, as required. □

The proof of the converse part is more formidable and is momentarily delayed. To give the reader a little more feeling for zigzag manipulation, we include the following surprising result.

Result 1.4.6 ([54]). Let Z be a zigzag in S over U with value d and spine $a_0,\ a_1, \ldots, a_{2m}$. If Z' be another zigzag in S over U with the same spine, then the value of Z' is also d.

Two such zigzags are, therefore, called equivalent.

Proof. Suppose Z is given by $a_0t_1 = y_1a_1t_1 = \cdots = y_ma_{2m}$, while Z' is given by $a_0t_1' = y_1'a_1t_1' = \cdots = y_m'a_{2m}$, with the appropriate equalities as given by Result 1.4.5 holding for both. Now, we have

$$d = a_0y_1 = y_1'a_1t_1 = y_1'a_2t_2 = \cdots = y_m'a_{2m} = \text{the value of } Z',$$

as required. □

We now prepare for a proof of the Zigzag Theorem. If M is a set and S is a semigroup with identity 1, we say that M is a *right S-system* if there is a mapping

$(x,s) \to xs$ from $M \times S$ into S such that $(xs)t = x(st)$ $(x \in M;\ s,t \in S)$ and $x1 = x$ $(x \in M)$. A *left S-system* is defined dually. If S and T are semigroups with identity, we say that M is an *(S,T)-bisystem* if it is a left S-system, a right T-system and for all $s \in S,\ t \in T$ and $x \in M,\ (sx)t = s(xt)$.

Let M be a right S-system and N be a left S-system. Let τ be the equivalence relation on $M \times N$ generated by $\{((xs,y),(x,sy)) : x \in M,\ s \in S,\ y \in N\}$. We denote $(M \times N)/\tau$ by $M \otimes_S N$ and call as the *tensor product* over S of two S-systems. The equivalence class $(x,y)\tau$ will be denoted by $x \otimes y$. Note that $xs \otimes y = x \otimes sy$ $(x \in M,\ s \in S,\ y \in N)$.

Observe that if M is a (T,S)-bisystem and N is an (S,U)-bisystem, then $M \otimes_S N$ becomes a (T,U)-bisystem if we define $t(x \otimes y) = tx \otimes y,\ (x \otimes y)u = x \otimes yu$, for $t \in T,\ u \in U,\ x \otimes y \in M \otimes_S N$.

If P and Q are right S-systems, we say that a map $\alpha : P \longrightarrow Q$ is a right *S-system morphism* if for every $x \in P$ and $s \in S$, $(xs)\alpha = (x\alpha)s$. Similar definitions apply to left *S-system* and *(S,T)-bisystem* morphisms.

Next, suppose that U is a subsemigroup of a semigroup S and let $S^{(1)}$ be the semigroup obtained from S by adjoining an identity element 1 whether it has 1 or not, and let $U^{(1)} = U \cup \{1\}$; then $U^{(1)}$ is a subsemigroup of $S^{(1)}$. We may, then, clearly form $S^{(1)} \otimes_{U^{(1)}} S^{(1)}$ and let $A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}$.

Result 1.4.7 ([51, Chapter VII Theorem 2.5]). If U is a subsemigroup of a semigroup S and $d \in S$, then $d \in Dom(U,S)$ if and only if $d \otimes 1 = 1 \otimes d$ in $A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}$.

Proof. Suppose that $d \in S$ and $d \otimes 1 = 1 \otimes d$ in A. The tensor product A is $(S^{(1)} \otimes S^{(1)})/\tau$, where τ is the equivalence relation on $S^{(1)} \times S^{(1)}$ generated by

$$T = \{((xu,y),\ (x,uy)) : x,\ y \in S^{(1)},\ u \in U^{(1)}\}.$$

Let R be a semigroup and let $\beta, \gamma : S \longrightarrow R$ be morphisms coinciding on U. We can regard β and γ as morphisms from $S^{(1)}$ into $R^{(1)}$ coinciding on $U^{(1)}$ by defining

$1\beta = 1\gamma = 1$. Define $\psi : S^{(1)} \times S^{(1)} \longrightarrow R^{(1)}$ by:

$$(x, y)\psi = (x\beta)(y\gamma), \ ((x, y) \in S^{(1)} \times S^{(1)}).$$

It may be easily checked that $T \subseteq \psi \circ \psi^{-1}$. Since $\psi \circ \psi^{-1}$ is an equivalence relation, the map $\chi : A \longrightarrow R^{(1)}$ denoted by $(x \otimes y)\chi = (x\beta)(y\gamma)$, $(x \otimes y \in A)$ is indeed well defined. But, now $(d \otimes 1)\chi = (1 \otimes d)\chi$; that is, $d\beta = d\gamma$ and so $d \in Dom(U, S)$.

To prove the converse, we regard the tensor product A as an $(S^{(1)}, S^{(1)})$-bisystem by defining

$$s(x \otimes y) = sx \otimes y, \quad (x \otimes y)s = x \otimes ys \quad (s, \ x, \ y \in S^{(1)}).$$

Let $(Z(A), \ +)$ be the free abelian group on A. The abelian group $Z(A)$ inherits an $(S^{(1)}, S^{(1)})$-bisystem structure from A if we define

$$s(\Sigma z_i a_i) = \Sigma z_i (s a_i), \quad (\Sigma z_i a_i)s = \Sigma z_i (a_i s),$$

for all $s \in S^{(1)}$ and $\Sigma z_i a_i \in Z(A)$. Observe that, for $x, \ y \in Z(A)$ and $s \in S^{(1)}$, we have

$$s(x + y) = sx + sy, \quad (x + y)s = xs + ys. \tag{1.4.2}$$

Next, we define a binary operation on $S^{(1)} \times Z(A)$ by

$$(p, x)(q, y) = (pq, py + xq). \tag{1.4.3}$$

Using the statements labeled (1.4.2) and (1.4.3), one verifies that this operation makes $S^{(1)} \times Z(A)$ a semigroup with identity $(1, 0)$.

We now consider two homomorphisms β and γ from $S^{(1)}$ into $S^{(1)} \times Z(A)$ and show that $\beta|U = \gamma|U$. Define β by $s\beta = (s, 0)$ $(s \in S^{(1)})$. Then, clearly, β is a morphism. Also define γ by $s\gamma = (s, s(1 \otimes 1) - (1 \otimes 1)s)$ $(s \in S^{(1)})$. To show that γ is a morphism, we denote $1 \otimes 1$ by a. Now using the statements (1.4.2) and (1.4.3), we verify that $(s, sa - as)(t, ta - at) = (st, s(ta - at) + (sa - as)t) = (st, (st)a - a(st))$. If $u \in U^{(1)}$, then

$$u(1 \otimes 1) = u \otimes 1 = 1u \otimes 1 = 1 \otimes u1 = (1 \otimes 1)u,$$

and so $u\beta = u\gamma$. Removing identities gives two morphisms β and γ from S into $S \times Z(A)$ such that $u\beta = u\gamma$ for all $u \in U$. If $d \in Dom(U, S)$, we must therefore have that $d\beta = d\gamma$; that is, $d \otimes 1 = 1 \otimes d$, as required. $\square$

We may now complete the proof of the Zigzag Theorem. Take $d \in Dom(U, S)$. By Result 1.4.3, we have that $d \otimes 1 = 1 \otimes d$ in the tensor product $A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}$. Hence, the pair $(1, d)$ and $(d, 1)$ are connected by a finite sequence of steps of the form

$$(xu, y) \to (x, uy), \tag{1.4.4}$$

or of the form

$$(x, uy) \to (xu, y). \tag{1.4.5}$$

If we have two successive steps

$$(xu, y) \to (x, uy) = (zv, uy) \to (z, vuy),$$

of the first type, we may achieve the same effect with a single step of this type:

$$(xu, y) = (zvu, y) \to (z, vuy).$$

A similar remark applies to the other case. Consequently, we may assume that steps of the two types occur alternately in the sequence connecting $(1, d)$ to $(d, 1)$.

The first and last steps must have the form

$$(1, d) = (1, uy) \to (u, y) \text{ and } (x, u) \to (xu, 1) = (d, 1) \text{ respectively.}$$

Hence the statement that $1 \otimes d = d \otimes 1$ is equivalent to the statement that $(1, d)$ is connected to $(d, 1)$ by a sequence of the steps as follows:

$$\begin{aligned}
(1, d) &= (1, a_0 t_1) \to (a_0, t_1) \\
&= (y_1 a_1, t_1) \to (y_1, a_1 t_1) \\
&= (y_1, a_2 t_2) \to (y_1 a_2, t_2) \\
&\quad \vdots \\
&= (y_i a_{2i-1}, t_i) \to (y_i, a_{2i-1} t_i)
\end{aligned}$$

$$= \quad (y_i, a_{2i}t_{i+1}) \to (y_i a_{2i}, t_{i+1})$$

$$\vdots$$

$$= \quad (y_m, a_{2m}) \to (y_m a_{2m}, 1) = (d, 1)$$

where $a_0, \ldots, a_{2m} \in U^{(1)}$, $y_1, \ldots, y_m$, $t_1, \ldots, t_m \in S^{(1)}$; and where $d = a_0 t_1$, $a_0 = y_1 a_1$, $a_{2m-1}t_m = a_{2m}$, $y_m a_{2m} = d$ and $a_{2i-1}t_i = a_{2i}t_{i+1}$, $y_i a_{2i} = y_{i+1}a_{2i+1}$ $(i = 1, 2, \ldots, m-1)$.

Without loss we may assume that each $a_i \in U$, since a transition of type (1.4.4) or of type (1.4.5) with $a = 1$ may be deleted. If any $y_i = 1$, let y_k be the last y_i that is equal to 1, then we have a subsequence of the sequence above as follows:

$$(1, d) \to \cdots \to (1, a_{2k}t_{k+1}) \text{ (but ending in } (1, a_{2m}) \text{ when } k = m).$$

Note that, if (p, q) and (r, s) are connected by steps of the form (1.4.4) and (1.4.5) then $pq = rs$. In the present instance this gives $d = a_{2k}t_{2k+1}$ (or $d = a_{2m}$); hence this sequence merely connects $(1, d)$ to $(d, 1)$ and so may be deleted. What remains is a sequence in which no y_i is 1.

A dual argument now ensures that we may construct the sequence from $(1, d)$ to $(d, 1)$ so that no t_i is 1. This completes the proof of the Zigzag Theorem. $\square$

In whatever follows, we shall refer to equations (1.4.1) as the zigzag equations.

Remark 1.4.8. The above Zigzag Theorem is also valid in the category of all commutative semigroups (Howie and Isbell [54]).

Dominions is also characterized in terms of special semigroup amalgams.

Result 1.4.9([42, Theorem 4.2.4]. Let U be a subsemigroup of S. Then d is in $Dom(U, S)$ if and only if for every weak embedding of the special amalgam $[S, S'; U]$ into some semigroup W by monomorphisms $\lambda : S \longrightarrow W$, $\bar{\lambda} : S' \longrightarrow W$, we have

$d\lambda = d\bar{\lambda}$.

Remark 1.4.10. From Result 1.4.9, we see that *a semigroup U is closed in S if and only if the special amalgam $[\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is embeddable, whereupon it follows that U is a special amalgamation base if and only if U is absolutely closed.* what is even more, results about amalgams may have corollaries for epis: every inverse semigroup is an amalgamation base implies that inverse semigroups are absolutely closed: every amalgam of inverse semigroups is strongly embeddable in an inverse semigroup implies that inverse semigroups are absolutely closed in the category of all inverse semigroups for instance.

The result due to Isbell [57], which provided a very interesting characterization of dominions in terms of special semigroup amalgams, is as follows:

Result 1.4.11 ([51, Chapter VII, Theorem 2.2]). Let U be a subsemigroup of a semigroup S. let S' be a semigroup disjoint from S and let $\alpha : S \longrightarrow S'$ be an isomorphism. Let $P = S *_U S'$, the free product of the amalgam

$$\mathcal{U} = [\{S, S'\};\ U\ ; \{i, \alpha|U\}],$$

where i is the inclusion mapping of U into S, and let $\mu,\ \mu'$ be the natural monomorphisms from $S,\ S'$ into P respectively. Then

$$(S\mu \cap S'\mu')\mu^{-1} = Dom(U, S).$$

In [80], Scheiblich had shown, by zigzag manipulations, that the class of all normal bands was closed. We provide the proof of this result as we have generalized this result to some categories of bands.

Result 1.4.12 ([80, Theorem 4.1]). Let B be a normal band and let A be subband of B. Then $Dom(A, B) = A$.

Proof. Take any $d \in Dom(A, B) \setminus A$. As $d \in Dom(A, B) \setminus A$, we may let, by Result 1.4.5, (1.4.1) be a zigzag of length m over U with value d. Now

$$
\begin{aligned}
d &= d^m \\
&= \prod_{i=1}^{m}(x_i a_{2i-1} y_i) \\
&= (x_1 a_1)(a_1 y_1)\prod_{i=2}^{m}(x_i a_{2i-1} y_i) \\
&= (x_1 a_1)(a_2 y_2)\prod_{i=2}^{m}(x_i a_{2i-1} y_i) \\
&= (x_1 a_1)(a_2)\prod_{i=2}^{m}(x_i a_{2i-1} y_i) \\
&= (x_1 a_1 a_2)(x_2 a_3)(a_3 y_2)\prod_{i=3}^{m}(x_i a_{2i-1} y_i) \\
&= (x_1 a_1 a_2)(x_2 a_3)(a_4 y_3)\prod_{i=3}^{m}(x_i a_{2i-1} y_i) \\
&= (x_1 a_1 a_2)(x_2 a_3 a_4)\prod_{i=3}^{m}(x_i a_{2i-1} y_i) \\
&= [\prod_{i=1}^{2}(x_i a_{2i-1} a_{2i})][\prod_{i=3}^{m}(x_i a_{2i-1} y_i)] \\
&\vdots \\
&= [\prod_{i=1}^{m-1}(x_i a_{2i-1} a_{2i})](x_m a_{2m-1})(a_{2m-1} y_m) \\
&= [\prod_{i=1}^{m-1}(x_i a_{2i-1} a_{2i})](x_{m-1} a_{2m-2})(a_{2m}) \\
&= [\prod_{i=1}^{m-1}(x_i a_{2i-1} a_{2i})][\prod_{i=m-1}^{m} a_{2i}] \\
&= [\prod_{i=1}^{m-2}(x_i a_{2i-1} a_{2i})](x_{m-1} a_{2m-3} a_{2m-2})[\prod_{i=m-1}^{m} a_{2i}]
\end{aligned}
$$

$$
\begin{aligned}
&= [\prod_{i=1}^{m-2}(x_i a_{2i-1} a_{2i})](x_{m-2} a_{2m-4})[\prod_{i=m-1}^{m} a_{2i}] \\
&= [\prod_{i=1}^{m-2}(x_i a_{2i-1} a_{2i})][\prod_{i=m-2}^{m} a_{2i}] \\
&\vdots \\
&= (x_1 a_1)(a_2)\prod_{i=2}^{m} a_{2i} \\
&= a_0 a_2 \prod_{i=2}^{m} a_{2i} \\
&= \prod_{i=0}^{m} a_{2i} \in A
\end{aligned}
$$

as required. $\square$

CHAPTER 2

ON CLOSED SEMIGROUPS

§ 2.1. INTRODUCTION

The present chapter envisages several results on closed semigroups. In 1962, Howie [44] showed that the amalgam $[S, T;\ U]$ is embeddable if U is almost unitary in S and $V = (U\phi)$ is almost unitary in T which implied that almost unitary subsemigroup of a semigroup is closed in the containing semigroup. In Section 2.2, we provide, by using zigzag manipulations, new and direct proof of this result and show that the special semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is embeddable if U is almost unitary in S.

In Section 2.3, we provide, by zigzag manipulations, a new and direct algebraic proof of [78, Corollary 6.5] and show that the special semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is embeddable if U is quasi-unitary in S. This generalizes the result of Section 2.2 which shows that the special semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is embeddable if U is almost unitary in S.

In Section 2.4, we find some closed classes of bands. First, we prove that the class of all left[right] regular bands is closed. Then we extend this result by showing that the class of all left[right] quasinormal bands is closed. We also show that the class of all $WL[WR]$ regular bands is closed and that the class of all left[right] regular bands is closed in the class of all regular bands. Lastly, we generalize this result by establishing that the class of all left[right] regular bands is closed in the class of all left[right] semiregular bands.

In Section 2.5, we show that the class of all semigroups satisfying the heterotypical identity $xyz = xz$, which contains the class of all rectangular bands, is closed.

In the last Section, we study some results based on closed semigroups satisfying the homotypical identity. First, we have shown that the class of all semigroups satisfying the homotypical identity $axa = ax[axa = xa]$, which contains the class of all left[right] regular bands, is closed. Next, we generalize the result by showing that the class of all semigroups satisfying the homotypical identity $axy = axay[yxa = yaxa]$, which contains the class of all left[right] quasinormal bands, is closed.

§ 2.2. ALMOST UNITARY SEMIGROUPS

Isbell characterized the notion of dominions in terms of special semigroup amalgams and shown that every amalgam is not strongly embeddable in general. see [51] for more details. Howie [44] had shown that if the core U of the amalgam $[S,T;\ U]$ is almost unitary in S and V is almost unitary in T, then the amalgam is embeddable. Infact, he proved the following:

Result 2.2.1 ([44, Theorem 3.3]). The amalgam $[S,T;\ U]$ is embeddable if U is almost unitary in S and $V(= U\phi)$ is almost unitary in T.

Definition 2.2.2 ([51, VII.3]). A subsemigroup U of a semigroup S is said to be *almost unitary* if there exist mappings $\lambda : S \longrightarrow S$, $\rho : S \longrightarrow S$ such that

$(AU1):\quad \lambda^2 = \lambda,\ \rho^2 = \rho;$

$(AU2):\quad \lambda(st) = (\lambda s)t, \quad (st)\rho = s(t\rho) \quad \forall s,t \in S;$

$(AU3):\quad \lambda(s\rho) = (\lambda s)\rho$ for every $s \in S$;

$(AU4):\quad s(\lambda t) = (s\rho)t \quad \forall s,t \in S;$

$(AU5):\quad \lambda \mid U{=}\rho \mid U = I_U;$

$(AU6)$: U is unitary in $\lambda S\rho$.

Theorem 2.2.3. Let U be a subsemigroup of a semigroup S. If U is almost-unitary in S, then U is closed in S.

Proof. To show that U is closed in S, take any $d \in Dom(U,S) \setminus U$. As d is in $Dom(U,S) \setminus U$, we may let, by Result 1.4.5, (1.4.1) be a zigzag of length m over U with value d. Since $a_0 \in U$, we have

$$\begin{aligned}
a_0 &= \lambda(a_0) && \text{(from (AU5))}\\
&= \lambda(x_1a_1) && \text{(from the zigzag equations)}\\
&= (\lambda(x_1))a_1 && \text{(from (AU2))}\\
&= (\lambda(x_1))\lambda(a_1) && \text{(from (AU5))}\\
&= (\lambda(x_1))\rho)a_1 && \text{(from (AU4))}\\
&= (\lambda x_1\rho)a_1 && \text{(from (AU3)).}
\end{aligned}$$

Therefore, $(\lambda x_1\rho)a_1 \in U$. Hence, by (AU6), $\lambda x_1\rho \in U$.

Now

$$\begin{aligned}
(\lambda x_1\rho)a_2 &= \lambda(x_1)(\lambda(a_2)) && \text{(from (AU4))}\\
&= (\lambda(x_1))a_2 && \text{(from (AU5))}\\
&= \lambda(x_1a_2) && \text{(from (AU2))}\\
&= \lambda(x_2a_3) && \text{(from the zigzag equations)}\\
&= (\lambda(x_2))a_3 && \text{(from (AU2))}\\
&= (\lambda(x_2))\lambda(a_3) && \text{(from (AU5))}\\
&= (\lambda x_2\rho)a_3 && \text{(from (AU4) and (AU3))}
\end{aligned}$$

$\Rightarrow \lambda x_2\rho \in U$ (by (AU6)).

Continuing this way, we have

$$\lambda x_m\rho \in U. \tag{2.2.1}$$

Now, we have

$$\begin{aligned}
d &= a_0y_1\\
&= \lambda(a_0)y_1 && \text{(by (AU5))}
\end{aligned}$$

$$
\begin{aligned}
&= \lambda(x_1a_1)y_1 && \text{(by the zigzag equations)}\\
&= (\lambda(x_1)a_1)y_1 && \text{(by (AU2))}\\
&= (\lambda(x_1)\lambda(a_1))y_1 && \text{(by (AU5))}\\
&= ((\lambda(x_1))\rho)a_1y_1 && \text{(by (AU4))}\\
&= (\lambda x_1\rho)a_1y_1 && \text{(by (AU3))}\\
&= (\lambda x_1\rho)a_2y_2 && \text{(by the zigzag equations)}\\
&= ((\lambda x_1\rho)a_2)y_2 &&\\
&= ((\lambda x_2\rho)a_3)y_2 && \text{(by (AU4),(AU5) and by the zigzag equations)}\\
&= (\lambda x_2\rho)a_3y_2 &&\\
&\vdots &&\\
&= (\lambda x_m\rho)a_{2m-1}y_m &&\\
&= (\lambda x_m\rho)a_{2m}.
\end{aligned}
$$

Since, by equation (2.2.1), $\lambda x_m\rho \in U$, we have $(\lambda x_m\rho)a_{2m} \in U$.

$\Rightarrow\ d \in U.$

Hence, $Dom(U, S) = U$, as required. □

§ 2.3. QUASI UNITARY SEMIGROUPS

In [78, Corollary 6.5], Renshaw had shown that the amalgam $[U; S_i]$ is strongly embeddable if U is quasi-unitary in each S_i which implied that a quasi-unitary subsemigroup is closed in any containing semigroup. We, now, provide a new and direct algebraic proof of this result, by zigzag manipulations, and show that the special semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ is embeddable if U is quasi-unitary in S. This generalizes [5, Theorem 2.1].

Definition 2.3.1 ([51, Exercises II.13]). A subsemigroup U of a semigroup S is called *right unitary [left unitary]* if

$$(\forall u \in U)(\forall s \in S)\ su \in U \Rightarrow s \in U[us \in U \Rightarrow s \in U].$$

A subsemigroup of a semigroup is said to be *unitary* if it is both left and right unitary.

Definition 2.3.2 ([53, Chapter 8]). A subsemigroup U of a semigroup S is said to be *quasi-unitary* if there is a (U^*, U^*)-morphism $\phi : S^* \to S^*$ such that

$$\phi^2 = \phi, \quad 1\phi = 1;$$
$$(\forall u \in U)(\forall s \in S)\ us \in U \Rightarrow s\phi \in U;$$
$$(\forall u \in U)(\forall s \in S)\ su \in U \Rightarrow s\phi \in U.$$

Trivially every unitary subsemigroup is quasi-unitary with ϕ as the identity map of S^*. Further if U is any quasi-unitary subsemigroup of S with $\phi : S^* \to S^*$ as a (U^*, U^*)-morphism, then for any $u \in U$

$$u\phi = (u1)\phi = u(1\phi) = u1 = u. \tag{2.3.1}$$

Therefore, for all $u \in U$ and $s \in S$, we have

$$(us)\phi = u(s\phi), \qquad (su)\phi = (s\phi)u. \tag{2.3.2}$$

Theorem 2.3.3. Let U be a quasi-unitary subsemigroup of a semigroup S. Let S' be a semigroup disjoint from S and let $\alpha : S \to S'$ be an isomorphism. If $P = S *_U S'$, the free product of the amalgam

$$\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}],$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P, then

$$S\mu \cap S'\mu' = U\mu;\ i.e.$$

the amalgam $\mathcal{U}$ is embeddable in a semigroup.

Proof. To prove the theorem, we essentially show that U is closed in S. So take any $d \in Dom(U, S) \setminus U$. As $d \in Dom(U, S) \setminus U$, we may let, by Result 1.4.5, (1.4.1) be a zigzag of minimal length m over U with value d. Since $a_0 \in U$, we have

$$
\begin{aligned}
a_0 &= a_0\phi && \text{(by equalities (2.3.1))} \\
&= (x_1a_1)\phi && \text{(from zigzag equations)} \\
&= (x_1\phi)a_1 && \text{(by equalities (2.3.2))} && (2.3.3) \\
(x_1\phi)a_2 &= (x_1a_2)\phi && \text{(by equalities (2.3.2))} \\
&= (x_2a_3)\phi && \text{(from zigzag equations)} \\
&= (x_2\phi)a_3 && \text{(by equalities (2.3.2))} && (2.3.4) \\
&\vdots \\
(x_{m-1}\phi)a_{2m-2} &= (x_{m-1}a_{2m-2})\phi && \text{(by equalities (2.3.2))} \\
&= (x_ma_{2m-1})\phi && \text{(from zigzag equations)} \\
&= (x_m\phi)a_{2m-1} && \text{(by equalities (2.3.2))} && (2.3.5) \\
(x_m\phi)a_{2m} &= (x_ma_{2m})\phi && \text{(by equalities (2.3.2))} \\
&= d\phi. && \text{(from zigzag equations)} && (2.3.6)
\end{aligned}
$$

Now, by equation (2.3.3), we have

$$(x_1\phi)\phi \in U \Rightarrow x_1\phi^2 \in U \Rightarrow x_1\phi \in U.$$

Similarly, from equations (2.3.4) and (2.3.5), we get $x_2\phi, x_m\phi \in U$.

Then, from equation (2.3.6), we have $d\phi = (x_m\phi)a_{2m} \in U$. (2.3.7)

Now

$$
\begin{aligned}
d &= a_0 y_1 \\
&= (a_0)\phi y_1 && \text{(by equalities (2.3.1))} \\
&= (x_1 a_1)\phi y_1 && \text{(by zigzag equations)} \\
&= (x_1 \phi) a_1 y_1 && \text{(by equalities (2.3.2))} \\
&= (x_1 \phi) a_2 y_2 && \text{(by zigzag equations)} \\
&= ((x_1 a_2)\phi) y_2 && \text{(by equalities (2.3.2))} \\
&= ((x_2 a_3)\phi) y_2 && \text{(by zigzag equations)} \\
&= (x_2 \phi) a_3 y_2 && \text{(by equalities (2.3.2))} \\
&\vdots \\
&= (x_{m-1}\phi) a_{2m-3} y_{m-1} \\
&= (x_{m-1}\phi) a_{2m-2} y_m && \text{(by zigzag equations)} \\
&= (x_{m-1} a_{2m-2})\phi y_m && \text{(by equalities (2.3.2))} \\
&= (x_m a_{2m-1})\phi y_m && \text{(by zigzag equations)} \\
&= (x_m \phi) a_{2m-1} y_m && \text{(by equalities (2.3.2))} \\
&= (x_m \phi) a_{2m} && \text{(by zigzag equations)}
\end{aligned}
$$

$$= \quad (x_m a_{2m})\phi \qquad \text{(by equalities (2.3.2))}$$

$$= \quad d\phi.$$

Therefore, from equation (8), $d = d\phi \in U$.

Hence $Dom(U, S) = U$, as required.

§ 2.4. CLASSES OF BANDS

In [80], Scheiblich had shown, by using zigzag manipulations, that the class of all normal bands is closed. We extend this result for the class of all left[right] regular bands and show, by using zigzag manipulations, that the class of all left[right] regular bands is closed.

Now we recall some definitions of various types of bands that we need to describe our forthcoming results. A band B is said to be

(1) left[right] semiregular band if $efg = efgegfg[gfe = gfgegfe]$ $\forall$ $e, f, g \in B$;

(2) regular band if $efge = efege$ $\forall$ $e, f, g \in B$;

(3) left[right] seminormal band if $efg = efgeg[gfe = gegfe]$ $\forall$ $e, f, g \in B$;

(4) left[right] quasinormal band if $efg = efeg[gfe = gefe]$ $\forall$ $e, f, g \in B$;

(5) WL[WR]- regular band if $efg = efege[fge = efege]$ $\forall$ $e, f, g \in B$;

(6) left[right] regular band if $ef = efe[fe = efe]$ $\forall$ $e, f, g \in B$.

For further details and others, one may refer to [72], [93] and [94].

Theorem 2.4.1. The class of all left regular bands is closed.

Proof. Let B be a left regular band and A be a subband of B; and let d is in

$Dom(A, B) \setminus A$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in B over A with value d of length m. Now

$$
\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 && \text{(by the zigzag equations)} \\
&= x_1 a_1 a_1 y_1 \\
&= x_1 a_1 a_2 y_2 && \text{(by the zigzag equations)} \\
&= x_1 a_1 x_1 a_2 y_2 && \text{(as B is a left regular band)} \\
&= x_1 a_1 x_2 a_3 y_2 && \text{(by the zigzag equations)} \\
&= x_1 a_1 x_2 a_3 a_3 y_2 \\
&= x_1 a_1 x_1 a_2 a_3 y_2 && \text{(by the zigzag equations)} \\
&= x_1 a_1 a_2 a_3 y_2 && \text{(as B is a left regular band)} \\
&= a_0 a_2 (a_3 y_2) && \text{(by the zigzag equations)} \\
&= (\prod_{i=0}^{1} a_{2i})(a_3 y_2) \\
&\vdots \\
&= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3} y_{m-1}) \\
&= a_0 a_2 a_4 \cdots a_{2m-4} a_{2m-3} y_{m-1}
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by the zigzag equations)}\\
&= x_1a_1x_1a_2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(as B is a left regular band)}\\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by the zigzag equations)}\\
&\vdots\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}y_m && \text{(as B is a left regular band)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}y_m && \text{(by the zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}y_m && \text{(as B is a left regular band)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_ma_{2m-2}x_ma_{2m-1}y_m && \text{(by the zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}a_{2m-1}y_m\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}a_{2m-1}y_m && \text{(by the zizag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}(a_{2m-1}y_m) && \text{(as B is a left regular band)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}(a_{2m}) && \text{(by the zigzag equations)}\\
&\vdots\\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by the zigzag equations)}\\
&= x_1a_1x_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by the zigzag equations)}\\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(as B is a left regular band)}\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by the zigzag equations)}
\end{aligned}
$$

$$= \prod_{i=0}^{m} a_{2i} \in A.$$

$\Rightarrow$ $d \in A$. Therefore, $Dom(A, B) = A$. Hence the theorem is proved. $\square$

Dually, we may prove the following:

Theorem 2.4.2. The class of all right regular bands is closed. $\square$

In the following, we extend Theorem 2.4.1 for the class of all left [right] quasinormal bands and show that the special semigroup amalgam $[\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ of a left [right] quasinormal band is embeddable in a left [right] quasinormal band .

Theorem 2.4.3. Let B be a left quasinormal band and let A be a subband of B. Then $Dom(A, B) = A$.

Proof. Take any $d \in Dom(A, B) \setminus A$. Then, by Result 1.4.5, we may let (1 4.1) be a zigzag in B over A with value d of minimal length m. Now

$$\begin{aligned}
d &= a_0 y_1 \\
&= a_0 a_0 y_1 && \text{(as A is a band)} \\
&= x_1 a_1 x_1 a_1 y_1 && \text{(by zigzag equations)} \\
&= x_1 a_1 x_1 a_2 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 a_3 y_2 && \text{(as A is a band)} \\
&= x_1 a_1 x_1 a_2 a_3 y_2 && \text{(by zigzag equations)}
\end{aligned}$$

$$
\begin{aligned}
&= x_1 a_1 a_2 a_3 y_2 \\
&= a_0 a_2 a_3 y_2 && \text{(by zigzag equations)} \\
&= a_0 a_2 (a_3 y_2) && \text{(by the zigzag equations)} \\
&= \prod_{i=0}^{1} a_{2i} (a_3 y_2) \\
&\vdots \\
&= \prod_{i=0}^{m-2} a_{2i} (a_{2m-3} y_{m-1}) \\
&= a_0 a_2 a_4 \cdots a_{2m-4} a_{2m-3} y_{m-1} \\
&= x_1 a_1 a_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_1 a_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m \\
&= x_1 a_1 x_2 a_3 a_4 \cdots a_{2m-4} a_{2m-2} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-2} a_{2m-4} a_{2m-2} y_m \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} a_{2m-2} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} x_{m-1} a_{2m-2} y_m \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} x_m a_{2m-1} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} x_m a_{2m-1} a_{2m-1} y_m && \text{(as A is a band)} \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} x_{m-1} a_{2m-2} (a_{2m-1} y_m) && \text{(by zigzag equations)}
\end{aligned}
$$

$$= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}(a_{2m-1}y_m)$$

$$= x_1a_1x_2a_3a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}(a_{2m-1}y_m) \qquad \text{(by zigzag equations)}$$

$$= x_1a_1x_1a_2a_4\cdots a_{2m-4}a_{2m-2}(a_{2m-1}y_m) \qquad \text{(by zigzag equations)}$$

$$= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}(a_{2m-1}y_m)$$

$$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}(a_{2m-1}y_m) \qquad \text{(by zigzag equations)}$$

$$= \prod_{i=0}^{m-1} a_{2i}(a_{2m-1}y_m)$$

$$= \prod_{i=0}^{m-1} a_{2i}(a_{2m}) \qquad \text{(by zigzag equations)}$$

$$= \prod_{i=0}^{m} a_{2i} \in A.$$

$\Rightarrow d \in A$. Hence $Dom(A, B) = A$.

Dually, we may prove the following:

Theorem 2.4.4. Let B be a right quasinormal band and let A be a subband of B. Then $Dom(A, B) = A$. □

The class of all WL[WR]-regular bands contains the class of all left [right] regular bands and is contained in the class of all regular bands. In the following theorem, we generalize Theorems 2.4.1 and 2.4.2, by establishing that all the class of all $WL[WR]$-regular bands is closed.

Theorem 2.4.5. The class $\mathcal{V}$ of all WL- regular bands is closed.

Proof. Take any $U, S \in \mathcal{V}$ with U a subsemigroup of S and let $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m. Now

$$\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_1 y_1 && \text{(since } a_1 \in S\text{)} \\
&= x_1 a_1 a_2 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 x_1 a_2 x_1 y_2 && \text{(since } x_1, a_1, a_2 \in S\text{)} \\
&= x_1 a_1 x_2 a_3 x_1 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 a_3 x_1 y_2 && \text{(since } a_3 \in S\text{)} \\
&= x_1 a_1 x_1 a_2 a_3 x_1 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_2 a_3 y_2 && \text{(since } x_1, a_1, a_2 a_3 \in S\text{)} \\
&= a_0 a_2 a_3 y_2 && \text{(by zigzag equations)} \\
&= \prod_{i=0}^{1} a_{2i} (a_3 y_2) \\
&\ \vdots \\
&= \prod_{i=0}^{m-2} a_{2i} (a_{2m-3} y_{m-1}) \\
&= a_0 a_2 a_4 \cdots a_{2m-4} (a_{2m-2} y_m) && \text{(by zigzag equations)}
\end{aligned}$$

$$
\begin{aligned}
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_1a_2x_1a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(since } x_1, a_1, a_2 \in S\text{)}\\
&= x_1a_1x_2a_3x_1a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_1x_2a_4x_2a_6\cdots a_{2m-4}a_{2m-2}y_m && \text{(since } x_2, a_3x_1, a_4 \in S\text{)}\\
&\ \vdots\\
&= x_1a_1x_2a_3x_1x_2a_4x_2a_6\cdots x_{m-2}a_{2m-4}a_{2m-2}y_m\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}x_{m-1}y_m && \text{(since } x_{m-1}, a_{2m-3}, a_{2m-2} \in S\text{)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}x_{m-1}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}a_{2m-1}x_{m-1}y_m && \text{(since } a_{2m-1} \in S\text{)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}a_{2m-1}x_{m-1}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}a_{2m-1}y_m && \text{(since } x_{m-1}, a_{2m-3}, a_{2m-2}a_{2m-1} \in S\text{)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&\ \vdots\\
&= x_1a_1x_2a_3x_1x_2a_4x_2a_6\cdots a_{2m-4}a_{2m-2}a_{2m}\\
&= x_1a_1x_2a_3x_1a_4a_6\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(since } x_2, a_3 \in S\text{)}\\
&= x_1a_1x_1a_2x_1a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}
\end{aligned}
$$

$= \quad x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}$ (since $x_1, a_1, a_2 \in S$)

$= \quad a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}$ (by zigzag equations)

$= \quad \prod_{i=0}^{m} a_{2i} \in U.$

$\Rightarrow \ d \in U$. Hence $Dom(U, S) = U$. □

We may also prove the following dual statement of the Theorem 2.4.5.

Theorem 2.4.6. The class $\mathcal{V}$ of all WR- regular bands is closed. □

As we have shown that the classes of all left [right] regular bands, left [right] quasinormal bands and WL[WR]-regular bands are closed, so, it is natural to ask about the closedness of the class of all regular bands.

We, now, provide a partial answer to the above question and show that the class of all left [right] regular bands is closed within the class of all regular bands.

Following lemma is very useful to prove our main result.

Lemma 2.4.7. Let U be a left regular band and S be any regular band such that U be a subband of S. If for $d \in Dom(U, S) \setminus U$ and (1.4.1) be a zigzag in S over U of minimal length m, then

$$(\prod_{i=0}^{m-1} a_{2i})y_m = (\prod_{i=0}^{m-1} a_{2i})a_{2m-1}(a_{2m-4}a_{2m-6}\cdots a_2a_0)(\prod_{i=0}^{m-1} a_{2i})y_m.$$

Proof. Now

$(\prod_{i=0}^{m-1} a_{2i})y_m$

$= \quad a_0a_2a_4\cdots a_{2m-4}a_{2m-2}y_m$

$= \quad x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}y_m$ (by zigzag equations)

$$
\begin{aligned}
&= (x_1a_1a_2)^2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a band)}\\
&= x_1a_1(a_2x_1x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}y_m\\
&= (x_1a_1)(a_2x_1a_2x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a regular band)}\\
&= (x_1a_1a_2)(x_1a_2x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)(x_2a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m && \text{(as } S \text{ is a band)}\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4^2)(x_2a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4a_4x_2)(a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4x_2a_4x_2)(a_3x_1a_1a_2a_4)\cdots a_{2m-4}a_{2m-2}y_m\\
& && \text{(as } S \text{ is a regular band)}\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)(x_3a_5x_2a_3x_1a_1a_2a_4a_6)\cdots a_{2m-4}a_{2m-2}y_m\\
& && \text{(by zigzag equations)}\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)(x_3a_5x_2a_3x_1a_1a_2a_4a_6)^2\cdots a_{2m-4}a_{2m-2}y_m\\
& && \text{(as } S \text{ is a band)}\\
&\vdots\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)\cdots(x_{m-1}a_{2m-3}x_{m-2}a_{2m-5}\cdots x_2a_3x_1a_1a_2a_4\cdots a_{2m-2})y_m\\
&= (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)\cdots(x_{m-1}a_{2m-3}\cdots x_2a_3x_1a_1a_2a_4\cdots a_{2m-2})^2y_m\\
& && \text{(as } S \text{ is a band)}
\end{aligned}
$$

$$= (x_1a_1a_2)\cdots(x_{m-1}a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))(x_{m-1}a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))y_m$$

$$= (x_1a_1a_2)\cdots x_{m-1}(a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-2} a_{2i}))a_{2m-2}x_{m-1}(a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))y_m$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1a_{2m-2}x_{m-1})z_2y_m$$

$$\text{(where } z_1 = a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-2} a_{2i}) \text{ and } z_2 = a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_{m-1}a_{2m-2}x_{m-1})z_2y_m \qquad \text{(as } S \text{ is a regular band)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_ma_{2m-1}x_{m-1})z_2y_m \qquad \text{(by zigzag equations)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_m(a_{2m-1}^2)x_{m-1})z_2y_m \qquad \text{(as } U \text{ is a band)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1x_{m-1}a_{2m-2}a_{2m-1}x_{m-1})z_2y_m \qquad \text{(by zigzag equations)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1a_{2m-2}a_{2m-1}x_{m-1})z_2y_m \qquad \text{(as } S \text{ is a regular band)}$$

$$= (x_1a_1a_2)\cdots(x_{m-1}z_1a_{2m-2}a_{2m-1}x_{m-1})(a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))y_m$$

$$\text{(since } z_2 = a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))$$

$$= (x_1a_1a_2)\cdots x_{m-1}z_1a_{2m-2}a_{2m-1}(x_{m-2}a_{2m-4}x_{m-2}a_{2m-5}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))y_m$$

$$\text{(by zigzag equations as } x_{m-1}a_{2m-3} = x_{m-2}a_{2m-4})$$

$$= (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}(x_{m-2}z_3a_{2m-1}x_{m-2}a_{2m-4}x_{m-2})z_3y_m$$

$$\text{(as } z_1 = a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-2} a_{2i}) \text{ and where}$$

$$z_3 = a_{2m-5}x_{m-3}a_{2m-7}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))$$

$$= \quad (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}(x_{m-2}z_3a_{2m-1}a_{2m-4}x_{m-2})z_3y_m$$

(as $x_{m-2}, z_3a_{2m-1}, a_{2m-4} \in S$ and S is a regular band)

$$= \quad (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}(x_{m-2}a_{2m-5}x_{m-3}z_4a_{2m-1}a_{2m-4}x_{m-2}a_{2m-5}x_{m-3})z_4y_m$$

(where $z_4 = a_{2m-7}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i})$ and as $z_3 = a_{2m-5}x_{m-3}a_{2m-7}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))$

$$= \quad (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}x_{m-3}a_{2m-6}(x_{m-3}z_4a_{2m-1}a_{2m-4}x_{m-3}a_{2m-6}x_{m-3})z_4y_m$$

(by zigzag equations)

$$= \quad (x_1a_1a_2)\cdots x_{m-1}a_{2m-3}x_{m-3}a_{2m-6}(x_{m-3}z_4a_{2m-1}a_{2m-4}a_{2m-6}x_{m-3})z_4y_m$$

(since $x_{m-3}, z_4a_{2m-1}a_{2m-4}, a_{2m-6} \in S$ and as S is a regular band)

$$\vdots$$

$$= \quad (x_1a_1a_2)\cdots(x_{m-1}a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))a_{2m-1}(a_{2m-4}a_{2m-6}\cdots a_2a_0(\prod_{i=0}^{m-1} a_{2i}))y_m$$

$$= \quad (x_1a_1a_2)\cdots z_6a_{2m-1}z_5y_m$$

(where $z_5 = a_{2m-4}a_{2m-6}\cdots a_2a_0(\prod_{i=0}^{m-1} a_{2i})$ and $z_6 = x_{m-1}a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))$

$$= \quad (x_1a_1a_2)(x_2a_3x_1a_1a_2a_4)\cdots z_6a_{2m-1}z_5y_m$$

$$= \quad (x_1a_1a_2x_1a_2x_1)a_1a_2a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(by zigzag equations)}$$

$$= \quad (x_1a_1a_2a_2x_1)a_1a_2a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(as } S \text{ is a regular band)}$$

$$= \quad (x_1a_1a_2)(x_1a_1a_2)a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(as } a_2 \in S\text{)}$$

$$= \quad (x_1a_1a_2)a_4\cdots z_6a_{2m-1}z_5y_m \qquad \text{(as } S \text{ is a band)}$$

$$
\begin{aligned}
&= a_0a_2a_4\cdots z_6a_{2m-1}z_5y_m && \text{(by zigzag equations)}\\
&= a_0a_2a_4\cdots(x_{m-1}a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))a_{2m-1}z_5y_m\\
& && (\text{as } z_6 = x_{m-1}a_{2m-3}\cdots x_2a_3(\prod_{i=0}^{m-1} a_{2i}))\\
&\vdots\\
&= (a_0a_2a_4\cdots a_{2m-4}a_{2m-2})a_{2m-1}z_5y_m\\
&= (a_0a_2a_4\cdots a_{2m-6}a_{2m-4}a_{2m-2})a_{2m-1}(a_{2m-4}a_{2m-6}\cdots a_2a_0(\prod_{i=0}^{m-1} a_{2i}))y_m\\
& && (\text{as } z_5 = a_{2m-4}a_{2m-6}\cdots a_2a_0(\prod_{i=0}^{m-1} a_{2i}))\\
&= (\prod_{i=0}^{m-1} a_{2i})a_{2m-1}(a_{2m-4}a_{2m-6}\cdots a_2a_0)(\prod_{i=0}^{m-1} a_{2i})y_m,
\end{aligned}
$$

as required. □

Theorem 2.4.8. Let $\mathcal{V}$ be the class of all left regular bands and $\mathcal{C}$ be the class of all regular bands. Then $\mathcal{V}$ is $\mathcal{C}$-closed.

Proof. Let U and S be a left regular band and a regular band respectively with U a subband of S. Take any $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.

Now

$$
\begin{aligned}
d &= a_0y_1\\
&= x_1a_1y_1 && \text{(by zigzag equations)}\\
&= x_1a_1a_1y_1\\
&= x_1a_1a_2y_2 && \text{(by zigzag equations)}
\end{aligned}
$$

$$
\begin{aligned}
&= (x_1a_1a_2)^2y_2 && \text{(as } S \text{ is a band)}\\
&= (x_1a_1a_2x_1)(a_1a_2y_2)\\
&= (x_1a_1x_1a_2x_1)a_1a_2y_2 && \text{(as } S \text{ is a regular band)}\\
&= x_1a_1x_2a_3x_1a_1a_2y_2 && \text{(by zigzag equations)}\\
&= (x_1a_1x_2a_3^2x_1)a_1a_2y_2\\
&= (x_1a_1x_1a_2a_3x_1)a_1a_2y_2 && \text{(by zigzag equations)}\\
&= (x_1a_1a_2a_3x_1)a_1a_2y_2 && \text{(as } S \text{ is a regular band)}\\
&= a_0a_2a_3a_0a_2y_2 && \text{(by zigzag equations)}\\
&= a_0a_2a_3y_2 && \text{(as } U \text{ is a left regular band)}\\
&= x_1a_1a_2a_4y_3 && \text{(by zigzag equations)}\\
&= (x_1a_1a_2)^2a_4y_3 && \text{(as } S \text{ is a band)}\\
&= (x_1a_1a_2x_1)a_1a_2a_4y_3\\
&= (x_1a_1x_1a_2x_1)a_1a_2a_4y_3 && \text{(as } S \text{ is a regular band)}\\
&= x_1a_1x_2a_3x_1a_1a_2a_4y_3 && \text{(by zigzag equations)}\\
&= x_1a_1(x_2a_3x_1a_1a_2a_4)^2y_3 && \text{(as } S \text{ is a band)}
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1(x_2a_3x_1a_1a_2a_4x_2)(a_3x_1a_1a_2a_4)y_3 && \\
&= x_1a_1(x_2a_3x_1a_1a_2x_2a_4x_2)(a_3x_1a_1a_2a_4)y_3 && \text{(as } S \text{ is a regular band)} \\
&= x_1a_1(x_2a_3x_1a_1a_2x_3a_5x_2)a_3x_1a_1a_2a_4y_3 && \text{(by zigzag equations)} \\
&= x_1a_1(x_2a_3x_1a_1a_2x_3a_5^2x_2)a_3x_1a_1a_2a_4y_3 && \text{(as } S \text{ is a band)} \\
&= x_1a_1(x_2a_3x_1a_1a_2x_2a_4a_5x_2)a_3x_1a_1a_2a_4y_3 && \text{(by zigzag equations)} \\
&= x_1a_1(x_2a_3x_1a_1a_2a_4a_5x_2)a_3x_1a_1a_2a_4y_3 && \text{(as } S \text{ is a regular band)} \\
&= x_1a_1x_1a_2(x_1a_1a_2a_4a_5x_1a_2x_1)a_1a_2a_4y_3 && \text{(by zigzag equations)} \\
&= x_1a_1x_1a_2(x_1a_1a_2a_4a_5a_2x_1)a_1a_2a_4y_3 && \text{(as } S \text{ is a regular band)} \\
&= (x_1a_1x_1a_2x_1)a_1a_2a_4a_5a_2a_0a_2a_4y_3 && \text{(by zigzag equations)} \\
&= (x_1a_1a_2x_1)a_1a_2a_4a_5a_2a_0a_2a_4y_3 && \text{(as } S \text{ is a regular band)} \\
&= (a_0a_2)^2a_4a_5a_2a_0a_2a_4y_3 && \text{(by zigzag equations)} \\
&= (a_0a_2)a_0a_2a_4a_5a_0a_2a_4y_3 && \text{(as } U \text{ is a band)} \\
&= a_0a_2a_4a_5a_0a_2a_4y_3 && \text{(as } U \text{ is a left regular band)} \\
&= a_0a_2a_4a_5y_3 && \text{(as } U \text{ is a left regular band)} \\
&\vdots && \\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-3}y_{m-1} &&
\end{aligned}
$$

$$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}y_m \qquad \text{(by zigzag equations)}$$

$$= (\prod_{i=0}^{m-1} a_{2i})y_m$$

$$= (\prod_{i=0}^{m-1} a_{2i})(a_{2m-1}a_{2m-4}a_{2m-6}\cdots a_2a_0)(\prod_{i=0}^{m-1} a_{2i})y_m \qquad \text{(by lemma 2.1)}$$

$$= (\prod_{i=0}^{m-1} a_{2i})w_1(\prod_{i=0}^{m-1} a_{2i})y_m \qquad \text{(where } w_1 = a_{2m-1}a_{2m-4}a_{2m-6}\cdots a_2a_0)$$

$$= (\prod_{i=0}^{m-1} a_{2i})w_1y_m \qquad (\text{as } \prod_{i=0}^{m-1} a_{2i}, w_1 \in U \text{ and } U \text{ is a left regular band})$$

$$= (\prod_{i=0}^{m-3} a_{2i})(a_{2m-4}a_{2m-2}a_{2m-1}a_{2m-4})a_{2m-6}\cdots a_2a_0y_m$$

$$(\text{as } w_1 = a_{2m-1}a_{2m-4}a_{2m-6}\cdots a_2a_0)$$

$$= (\prod_{i=0}^{m-3} a_{2i})(a_{2m-4}a_{2m-2}a_{2m-1})a_{2m-6}\cdots a_2a_0y_m$$

$$(\text{as } a_{2m-4}, a_{2m-2}a_{2m-1} \in U \text{ and } U \text{ is a left regular band})$$

$$= (\prod_{i=0}^{m-1} a_{2i})a_{2m-1}a_{2m-6}\cdots a_2a_0y_m$$

$$= (\prod_{i=0}^{m-4} a_{2i})(a_{2m-6}a_{2m-4}a_{2m-2}a_{2m-1}a_{2m-6})\cdots a_2a_0y_m$$

$$= (\prod_{i=0}^{m-4} a_{2i})(a_{2m-6}a_{2m-4}a_{2m-2}a_{2m-1})a_{2m-8}\cdots a_2a_0y_m$$

$$(\text{as } a_{2m-6}, a_{2m-4}a_{2m-2}a_{2m-1} \in U \text{ and } U \text{ is a left regular band})$$

$$= (\prod_{i=0}^{m-1} a_{2i})a_{2m-1}a_{2m-8}\cdots a_2a_0y_m$$

$$\vdots$$

$$= (\prod_{i=0}^{m-1} a_{2i})a_{2m-1}a_2a_0y_m$$

$$= a_0a_2(a_4 \cdots a_{2m-4}a_{2m-2}a_{2m-1})a_2a_0y_m$$

$$= a_0a_2(a_4 \cdots a_{2m-4}a_{2m-2}a_{2m-1})a_0y_m$$

(as $a_2, (a_4 \cdots a_{2m-4}a_{2m-2}a_{2m-1}) \in U$ and U is a left regular band)

$$= a_0(a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m-1})a_0y_m$$

$$= a_0(a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m-1})y_m$$

(as $a_0, (a_2a_4 \cdots a_{2m-4}a_{2m-2}a_{2m-1}) \in U$ and U is a left regular band)

$$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}(a_{2m-1}y_m)$$

$$= a_0a_2a_4 \cdots a_{2m-4}a_{2m-2}(a_{2m}) \qquad \text{(by zigzag equations)}$$

$$= \prod_{i=0}^{m} a_{2i} \in U$$

$\Rightarrow$ $d \in U$. Hence $Dom(U, S) = U$. □

Dually, we may prove the following:

Theorem 2.4.9. Let $\mathcal{V}$ be a class of right regular bands and $\mathcal{C}$ be a class of all regular bands. Then $\mathcal{V}$ is $\mathcal{C}$-*closed*. □

In the following, we extend Theorems 2.4.8 and 2.4.9 by establishing that the class of all left [right] regular bands is closed within the class of all left [right] semiregular bands.

Theorem 2.4.10. Let $\mathcal{V}$ be a class of left regular bands and $\mathcal{C}$ be the class of all left semiregular bands. Then $\mathcal{V}$ is $\mathcal{C}$-closed.

Proof. Let U and S be a left regular band and left semiregular band respectively with

U a subband of S and let $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.
Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)} \\
&= x_1a_1^2y_1 \\
&= x_1a_1a_2y_2 && \text{(by zigzag equations)} \\
&= (x_1a_1a_2x_1a_2a_1a_2)y_2 && \text{(as } S \text{ is a left semiregular band)} \\
&= (x_1a_1a_2x_2a_3a_1a_2)y_2 && \text{(by zigzag equations as } x_1a_2 = x_2a_3) \\
&= (x_1a_1a_2x_2a_3a_3a_1a_2)y_2 \\
&= x_1a_1a_2x_1a_2a_3a_1a_2y_2 && \text{(by zigzag equations as } x_2a_3 = x_1a_2) \\
&= x_1a_1a_2x_1a_2(a_3a_1a_2a_3)y_2 && \text{(as } U \text{ is left regular band)} \\
&= x_1a_1a_2x_2a_3a_1a_2a_3y_2 && \text{(by zigzag equations as } x_1a_2 = x_2a_3) \\
&= (x_1a_1a_2x_1a_2a_1a_2)a_3y_2 && \text{(by zigzag equations as } x_2a_3 = x_1a_2) \\
&= (x_1a_1a_2)a_3y_2 && \text{(as } S \text{ is a left semiregular band)} \\
&= a_0a_2(a_3y_2) && \text{(by zigzag equations as } x_1a_1 = a_0)
\end{aligned}$$

$$= (\prod_{i=0}^{1} a_{2i})(a_3 y_2)$$

$$\vdots$$

$$= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3} y_{m-1})$$

$$= x_1 a_1 a_2 a_4 \cdots a_{2m-4}(a_{2m-2} y_m) \qquad \text{(by zigzag equations)}$$

$$= x_1 a_1 a_2 x_1 a_2 a_1 a_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m \qquad \text{(since } x_1, a_1, a_2 \in S\text{)}$$

$$= (x_1 a_1 a_2)(x_2 a_3 a_1 a_2 a_4) \cdots a_{2m-4} a_{2m-2} y_m \quad \text{(by zigzag equations)}$$

$$= (x_1 a_1 a_2)(x_2 a_3 a_1 (\prod_{i=1}^{2} a_{2i}))(x_2 a_4 a_3 a_1 (\prod_{i=1}^{3} a_{2i})) \cdots a_{2m-4} a_{2m-2} y_m$$

$$\text{(since } x_2, a_3 a_1 a_2, a_4 \in S\text{)}$$

$$= (x_1 a_1 a_2)(x_2 a_3 a_1 (\prod_{i=1}^{2} a_{2i}))(x_3 a_5 a_3 a_1 (\prod_{i=1}^{3} a_{2i})) \cdots a_{2m-4} a_{2m-2} y_m$$

$$\text{(by zigzag equations)}$$

$$\vdots$$

$$= (x_1 a_1 a_2)(x_2 a_3 a_1 a_2 a_4) \cdots (x_{m-1} a_{2m-3} a_{2m-5} \cdots a_3 a_1 (\prod_{i=1}^{m-2} a_{2i}) a_{2m-2}) y_m$$

(by zigzag equations and as S is a left semiregular band)

$$= (x_1 a_1 a_2)(x_2 a_3 a_1 a_2 a_4) \cdots (x_{m-1} z_1 a_{2m-2}) y_m$$

$$\text{(where } z_1 = a_{2m-3} a_{2m-5} \cdots a_3 a_1 (\prod_{i=1}^{m-2} a_{2i})\text{)}$$

$$= (x_1 a_1 a_2)(x_2 a_3 a_1 a_2 a_4) \cdots (x_{m-1} z_1 a_{2m-2} x_{m-1} a_{2m-2} z_1 a_{2m-2}) y_m$$

$$\text{(since } x_{m-1}, z_1, a_{2m-2} \in S\text{)}$$

$$= (x_1a_1a_2)(x_2a_3a_1a_2a_4)\cdots(x_{m-1}z_1a_{2m-2}x_ma_{2m-1}z_1a_{2m-2})y_m$$
(by zigzag equations)

$$= (x_1a_1a_2)(x_2a_3a_1a_2a_4)\cdots(x_{m-1}z_1a_{2m-2}x_ma_{2m-1}z_1a_{2m-2}a_{2m-1})y_m$$
(as $a_{2m-1}, z_1, a_{2m-2} \in U$ and U is a left regular band)

$$= (x_1a_1a_2)(x_2a_3a_1a_2a_4)\cdots(x_{m-1}z_1a_{2m-2}x_ma_{2m-1}z_1a_{2m-2})a_{2m-1}y_m$$

$$= (x_1a_1a_2)(x_2a_3a_1a_2a_4)\cdots(x_{m-1}z_1a_{2m-2}x_{m-1}a_{2m-2}z_1a_{2m-2})a_{2m}$$
(by zigzag equations as $x_{m-1}a_{2m-2} = x_ma_{2m-1}$)

$$= (x_1a_1a_2)(x_2a_3a_1a_2a_4)\cdots(x_{m-1}z_1a_{2m-2})a_{2m} \qquad \text{(as } x_{m-1}, z_1, a_{2m-2} \in S)$$

$$= (x_1a_1a_2x_2a_3a_1a_2)a_4\cdots(x_{m-1}z_1a_{2m-2})a_{2m}$$

$$= (x_1a_1a_2)a_4\cdots(x_{m-1}z_1a_{2m-2})a_{2m} \qquad \text{(as } x_1, a_1, a_2 \in S)$$

$$= a_0a_2a_4\cdots(x_{m-1}z_1a_{2m-2})a_{2m} \qquad \text{(by zigzag equations as } x_1a_1 = a_0)$$

$$\vdots$$

$$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}$$

$$= \prod_{i=0}^{m} a_{2i} \in U.$$

$\Rightarrow d \in U.$

Hence $Dom(U, S) = U$. □

Dually, we may prove the following:

Theorem 2.4.11. Let $\mathcal{V}$ be the class of all right regular bands and $\mathcal{C}$ be the class of all right semiregular bands. Then $\mathcal{V}$ is $\mathcal{C}$-closed. $\square$

§ 2.5. HETEROTYPICAL IDENTITY

Theorem 2.5.1. Let S be a semigroup satisfying an identity $xyz = xz$ and U be a subsemigroup of S satisfying an identity $xyz = xz$, then U is closed in S.

Proof. Let us take $d \in Dom(U,S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.
Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)} \\
&= x_1a_2y_2 && \text{(by zigzag equations)} \\
&= x_1a_1a_2y_2 && \text{(since } x_1, a_1, a_2 \in S) \\
&= x_1a_1a_2a_3y_2 && \text{(since } a_2, a_3, y_2 \in S) \\
&= a_0a_2a_3y_2 && \text{(by zigzag equations)} \\
&= \prod_{i=0}^{1} a_{2i}(a_3y_2) \\
&\vdots \\
&= \prod_{i=0}^{m-2} a_{2i}(a_{2m-3}y_{m-1}) \\
&= a_0a_2\cdots a_{2m-4}(a_{2m-2}y_m) && \text{(by zigzag equations)} \\
&= a_0a_2\cdots a_{2m-4}a_{2m-2}y_m
\end{aligned}$$

$$= \quad a_0a_2 \cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m \qquad \text{(since } a_{2m-2}, a_{2m-1}, y_m \in S\text{)}$$

$$= \quad a_0a_2 \cdots a_{2m-4}a_{2m-2}a_{2m} \qquad \text{(by zigzag equations)}$$

$$= \quad \prod_{i=0}^{m} a_{2i} \in U.$$

$\Rightarrow \quad d \in U$. Hence $Dom(U, S) = U$. □

§ 2.6. HOMOTYPICAL IDENTITIES

Theorem 2.6.1. Let U and S be a semigroups satisfying the identity $xy = xyx$ and with U a subsemigroup of S. Then U is closed in S.

In order to prove our result, we need the following useful remark which may be well known.

Remark 2.6.2. Let S be a semigroup satisfying the identity $xy = xyx$. If $E(S)$ be the set of idempotents of S, then $S^2 \subseteq E(S)$.

Proof. Let us take $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m. Now

$$d \quad = \quad a_0y_1$$

$$= \quad x_1a_1y_1 \qquad \text{(by zigzag equations)}$$

$$= \quad x_1a_1x_1a_1y_1 \qquad \text{(since } x_1a_1 \in E(S)\text{)}$$

$$= \quad x_1a_1x_1a_2y_2 \qquad \text{(by zigzag equations)}$$

$$= \quad x_1a_1x_2a_3y_2 \qquad \text{(by zigzag equations)}$$

$$
\begin{aligned}
&= x_1a_1x_2a_3x_2a_3y_2 && (\text{since } x_2a_3 \in E(S)) \\
&= x_1a_1x_1a_2a_3y_2 && (\text{by zigzag equations}) \\
&= x_1a_1a_2a_3y_2 && (\text{since } x_1, a_1 \in S) \\
&= a_0a_2a_3y_2 && (\text{by zigzag equations}) \\
&= \prod_{i=0}^{1} a_{2i}(a_3y_2) \\
&\vdots \\
&= \prod_{i=0}^{m-2} a_{2i}(a_{2m-3}y_{m-1}) \\
&= a_0a_2\cdots a_{2m-4}(a_{2m-2}y_m) && (\text{by zigzag equations}) \\
&= x_1a_1a_2\cdots a_{2m-4}a_{2m-2}y_m && (\text{by zigzag equations}) \\
&= x_1a_1x_1a_2\cdots a_{2m-4}a_{2m-2}y_m && (\text{since } x_1, a_1 \in S) \\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}y_m && (\text{by zigzag equations}) \\
&= x_1a_1x_2a_3x_2a_4\cdots a_{2m-4}a_{2m-2}y_m && (\text{since } x_2, a_3 \in S) \\
&\vdots \\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}y_m \\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}y_m && (\text{by zigzag equations}) \\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}y_m && (\text{since } x_{m-1}, a_{2m-3} \in S) \\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}y_m && (\text{by zigzag equations})
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}x_ma_{2m-1}y_m && (\text{since } x_ma_{2m-1}\in E(S))\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}a_{2m-1}y_m && (\text{since } x_m, a_{2m-1}\in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}a_{2m} && (\text{by zigzag equations})\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}a_{2m} && (\text{since } x_{m-1}, a_{2m-3}\in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m} && (\text{by zigzag equations})\\
&\vdots\\
&= x_1a_1x_2a_3x_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}\\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && (\text{since } x_2, a_3\in S)\\
&= x_1a_1x_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && (\text{by zigzag equations})\\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && (\text{since } x_1, a_1\in S)\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && (\text{by zigzag equations})\\
&= \prod_{i=0}^{m} a_{2i}\in U
\end{aligned}
$$

$\Rightarrow d\in U$. Hence $Dom(U,S)=U$. □

Dually, we may prove the following:

Theorem 2.6.3. Let $U$and S be a semigroups satisfying the identity $yx = xyx$ and with U a subsemigroup of S. Then U is closed in S. □

In [4], authors have shown that the class of all left[right] quasinormal bands are closed within the class of all left[right] quasinormal bands. In the following theorem, we generalize this result and show that the class $\mathcal{V}=[axy = axay]$ of all semigroups is closed.

Theorem 2.6.4. The class $\mathcal{V}=[axy = axay]$ of semigroups is closed.

Proof. Take any $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m. Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)} \\
&= x_1a_1x_1y_1 && \text{(since } x_1, a_1, y_1 \in S) \\
&= x_1a_1x_1a_1y_1 && \text{(since } a_1, x_1, y_1 \in S) \\
&= x_1a_1a_1y_1 && \text{(since } x_1, a_1, y_1 \in S) \\
&= x_1a_1a_2y_2 && \text{(by zigzag equations)} \\
&= x_1a_1x_1a_2y_2 && \text{(since } x_1, a_1 \in S) \\
&= x_1a_1x_2a_3y_2 && \text{(by zigzag equations)} \\
&= x_1a_1x_2a_3x_2a_3y_2 && \text{(since } x_2, a_3, y_2 \in S) \\
&= x_1a_1x_2a_3a_3y_2 && \text{(since } x_2, a_3 \in S) \\
&= x_1a_1x_1a_2a_3y_2 && \text{(by zigzag equations)}
\end{aligned}$$

$$
\begin{aligned}
&= x_1 a_1 a_2 a_3 y_2 && \text{(since } x_1, a_1, a_2 \in S) \\
&= a_0 a_2 a_3 y_2 && \text{(by zigzag equations)} \\
&= \prod_{i=0}^{1} a_{2i}(a_3 y_2) \\
&\vdots \\
&= \prod_{i=0}^{m-2} a_{2i}(a_{2m-3} y_{m-1}) \\
&= a_0 a_2 a_4 \cdots a_{2m-4}(a_{2m-2} y_m) && \text{(by zigzag equations)} \\
&= x_1 a_1 a_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_1 a_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m && \text{(since } x_1, a_1, a_2 \in S) \\
&= x_1 a_1 x_2 a_3 a_4 \cdots a_{2m-4} a_{2m-2} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots a_{2m-4} a_{2m-2} y_m && \text{(since } x_2, a_3, a_4 \in S) \\
&\vdots \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-2} a_{2m-4} a_{2m-2} y_m \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} a_{2m-2} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} x_{m-1} a_{2m-2} y_m && \text{(since } x_{m-1}, a_{2m-3}, a_{2m-2} \in S) \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} x_m a_{2m-1} y_m && \text{(by zigzag equations)} \\
&= x_1 a_1 x_2 a_3 x_2 a_4 \cdots x_{m-1} a_{2m-3} x_m a_{2m-1} x_m y_m && \text{(since } x_m, a_{2m-1}, y_m \in S)
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}x_ma_{2m-1}y_m && (\text{since } a_{2m-1}, x_m, y_m \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}a_{2m-1}y_m && (\text{since } x_m, a_{2m-1}, a_{2m-1}y_m \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}a_{2m-1}y_m && (\text{by zigzag equations})\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}a_{2m-1}y_m && (\text{since } x_{m-1}, a_{2m-3}, a_{2m-2} \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m-1}y_m && (\text{by zigzag equations})\\
&\vdots\\
&= x_1a_1x_2a_3x_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m\\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m && (\text{since } x_2, a_3x_1, a_4 \in S)\\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m && (\text{since } x_2, a_3, a_4 \in S)\\
&= x_1a_1x_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && (\text{by zigzag equations})\\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && (\text{since } x_1, a_1, a_2 \in S)\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && (\text{by zigzag equations})\\
&= \prod_{i=0}^{m} a_{2i} \in U.
\end{aligned}
$$

$\Rightarrow$ $d \in U$. Hence $Dom(U, S) = U$. □

Dually we have the following :

Theorem 2.6.5. The class $\mathcal{V}$=$[yxa = yaxa]$ of semigroups is closed. □

CHAPTER 3

ON SATURATED SEMIGROUPS

§ 3.1. INTRODUCTION

An effort has been made to identify the saturated classes of algebras in semigroup theory, ring theory, and elsewhere [8]. For example, B.J. Gardner [20, Theorem 2.10] has shown that any class of regular rings is saturated in the class of all rings whereas P.M. Higgins [33, Corollary 4], in contrast, recently has shown that not all regular semigroups are saturated. However, in [30], P.M. Higgins has shown that any class of generalized inverse semigroup is saturated. A necessary condition for any semigroup variety to be saturated is that it admits an identity, not a permutation identity, of which at least one side has no repeated variable [36, Theorem 6]. In [29, Theorem 1], Hall and Jones have shown that any class of completely semisimple semigroups with a finite number of J-classes (where J is the usual Green's relation) is saturated. In particular, any class of finite regular semigroup is saturated. P.M. Higgins [35, Theorem 10] has extended the result of Howie and Isbell [54, Theorem 3.2] that any class of finite commutative semigroup is saturated. He proves that any finite permutative semigroup is saturated. However, the determination of all saturated varieties of semigroups is still an open problem, although the question has been settled for commutative varieties (Higgins [31]; Khan [58]) and for heterotypical varieties, which are those varieties not containing the variety of all semilattices [36]. However, there are semigroups of each of the following types which are not saturated; commutative cancellative semigroups (the injection of the natural numbers into the integers provides an example), subsemigroups of finite inverse semigroups [57], commutative periodic semigroups [31] and bands, as Trotter [89] has constructed a band with a properly epimorphically embedded subband.

In this chapter, we prove some results on saturated semigroups. In Section 3.2, we show that the class of all inflations of Clifford semigroups is saturated. We, then, in Section 3.3, show that a subclass of the class of all semigroups satisfying the identity $ax = axa[xa = axa]$ is saturated. Finally, in Section 3.4, we show that the class of all quasi-commutative semigroups satisfying a nontrivial identity I such that at least one side of I has no repeated variable is saturated. This generalizes the long known result of Khan [58], which shows that the class of all commutative semigroups satisfying a nontrivial identity I such that at least one side of I has no repeated variable is saturated.

§ 3.2. INFLATION OF A CLIFFORD SEMIGROUP

In 1967, Howie and Isbell [54] have shown that the class of all inverse semigroups is absolutely closed which implied that the class of all Clifford semigroups is absolutely closed. Since the class of all Clifford semigroups is contained in the class of all inflations of Clifford semigroups, so this raises a question whether inflations of Clifford semigroups are absolutely closed or not? Towards this goal, we have established that the class of all inflations of Clifford semigroups is saturated. However, the question of whether the class of all inflations of Clifford semigroups is absolutely closed or not is still remains open.

Definition 3.2.1. A semigroup S is called an inflation of a semigroup T if T is a subsemigroup of S and there is a mapping ϕ of S into T such that $\phi(x) = x$ for $x \in T$ and $xy = \phi(x)\phi(y)$ for $x, y \in S$.

Remark 3.2.2. ([88]). A semigroup S is an *inflation of a Clifford semigroup* if and only if S^2 is a Clifford semigroup.

Result 3.2.3 ([61, Result 3]). Let U be any subsemigroup of a semigroup S and let $d \in Dom(U, S) \setminus U$. If (1.4.1) is a zigzag of minimum length m over U with value d, then $y_j,\ t_j \in S \setminus U$ for $j = 1,\ 2, \ldots, m$.

In the following results, let U and S be any semigroups with U dense in S.

Result 3.2.4 ([61, Result 4]). For any $d \in S \setminus U$ and k any positive integer, if (1.4.1) is a zigzag of minimum length m over U with value d, then there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1 b_2 \cdots b_k d_k$.

Theorem 3.2.5. Any inflation of a Clifford semigroup is saturated.

Proof. Suppose U be any inflation of a Clifford semigroup. Suppose to the contrary that U is not saturated. Then, there exists a semigroup S containing U properly and such that $Dom(U, S) = S$. Take any $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m. As (1.4.1) is of minimal length, by Result 3.2.3, $x_i,\ y_i \in S \setminus U$ for all $i = 1, 2, \ldots, m$. Therefore, by Result 3.2.4,

$$x_i = z_i u_i,\ y_i = v_i t_i \tag{3.2.1}$$

for some $z_i,\ t_i \in S \setminus U;\ \ u_i,\ v_i \in U\ (i = 1, 2, \ldots, m)$.
Now

$$\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 && \text{(by zigzag equations)} \\
&= (z_1 u_1) a_1 y_1 && \text{(by equalities (3.2.1) as } x_1 \in S \setminus U) \\
&= z_1 (u_1 a_1)(u_1 a_1)'(u_1 a_1) y_1 && \text{(since } u_1 a_1 \text{ is regular)} \\
&= x_1 a_1 (u_1 a_1)(u_1 a_1)'(u_1 a_1) y_1 && \text{(by equalities (3.2.1) as } x_1 \in S \setminus U) \\
&= x_1 a_1 (u_1 a_1)' u_1 a_2 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 (u_1 a_1)' u_1 a_2 v_2 t_2 && \text{(by equalities (3.2.1) as } y_2 \in S \setminus U) \\
&= x_1 a_1 (u_1 a_1)' u_1 (a_2 v_2)(a_2 v_2)'(a_2 v_2) t_2 \\
&= x_1 (a_2 v_2)(a_2 v_2)' a_1 (u_1 a_1)' u_1 (a_2 v_2) t_2
\end{aligned}$$

(since $(a_2 v_2)(a_2 v_2)'$ is idempotent and is in the center of U)

$$
\begin{aligned}
&= x_2a_3v_2(a_2v_2)'a_1(u_1a_1)'u_1(a_2v_2)t_2 && \text{(by zigzag equations)} \\
&= z_2u_2a_3v_2(a_2v_2)'a_1(u_1a_1)'u_1(a_2v_2)t_2 \quad \text{(since } x_2 \in S\text{)} \\
&= z_2(u_2a_3)(u_2a_3)'(u_2a_3)v_2(a_2v_2)'a_1(u_1a_1)'u_1(a_2v_2)t_2 \\
& && \text{(since } u_2a_3 \text{ is regular element of } U\text{)} \\
&= x_2a_3v_2(a_2v_2)'a_1(u_1a_1)'u_1a_2(u_2a_3)'(u_2a_3)y_2 \\
& && \text{(since } (a_2v_2)(a_2v_2)' \text{ is idempotent and is in the center of } U\text{)} \\
&= x_2a_3(v_2(a_2v_2)'a_1(u_1a_1)'u_1a_2(u_2a_3)'u_2)(a_3y_2) && \text{(by zigzag equations)} \\
&= x_2a_3(w_2)(a_3y_2) && \text{(where } w_2 = (u_1a_1)'a_2v_2(a_2v_2)'u_1a_2(u_2a_3)'u_2 \in U\text{)} \\
&\vdots \\
&= x_{m-1}a_{2m-3}(w_{m-1})(a_{2m-3}y_{m-1}) && \text{(where } w_{m-1} = w_{m-2}w_{m-3}\cdots w_2 \in U\text{)} \\
&= x_{m-1}a_{2m-3}(w_{m-1})a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= x_{m-1}a_{2m-3}(w_{m-1})a_{2m-2}v_mt_m && \text{(by equalities (3.2.1) as } y_m \in S \setminus U\text{)} \\
&= x_{m-1}a_{2m-3}(w_{m-1})(a_{2m-2}v_m)(a_{2m-2}v_m)'(a_{2m-2}v_m)t_m \\
&= x_{m-1}(a_{2m-2}v_m)(a_{2m-2}v_m)'a_{2m-3}(w_{m-1})(a_{2m-2}v_m)t_m \\
& && \text{(since } (a_2v_2)(a_2v_2)' \text{ is idempotent and is in the center of } U\text{)} \\
&= x_ma_{2m-1}v_m(a_{2m-2}v_m)'a_{2m-3}(w_{m-1})(a_{2m-2}v_m)t_m && \text{(by zigzag equations)} \\
&= z_mu_ma_{2m-1}v_m(a_{2m-2}v_m)'a_{2m-3}(w_{m-1})(a_{2m-2}v_m)t_m \\
& && \text{(by equalities (3.2.1) as } y_m \in S \setminus U\text{)}
\end{aligned}
$$

$$= z_m(u_m a_{2m-1})(u_m a_{2m-1})'(u_m a_{2m-1})v_m(a_{2m-2}v_m)'a_{2m-3}(w_{m-1})(a_{2m-2}v_m)t_m$$

$$= x_m a_{2m-1}v_m(a_{2m-2}v_m)'a_{2m-3}(w_{m-1})a_{2m-2}(u_m a_{2m-1})'(u_m a_{2m-1})y_m$$

(since $(u_m a_{2m-1})'(u_m a_{2m-1})$ is idempotent and is in the center of U)

$$= x_{m-1}a_{2m-2}v_m(a_{2m-2}v_m)'a_{2m-3}(w_{m-1})a_{2m-2}(u_m a_{2m-1})'u_m a_{2m}$$

(by zigzag equations)

$$\vdots$$

$$= x_1 a_1(w_{m-1})a_{2m} \quad (\text{where } w_m = w_{m-1}w_{m-2}w_{m-3}\cdots w_2 \in U)$$

$$= a_0(w_{m-1})a_{2m} \in U$$

$\Rightarrow d \in U$, a contradiction.

Therefore, $Dom(U, S) \neq S$ and, hence, U is saturated. $\square$

§ 3.3. SEMIGROUP SATISFYING THE IDENTITY $ax = axa[xa = axa]$

In [2], authors have shown that the class of all semigroups satisfying the identity $ax = axa[xa = axa]$, which includes the class of all left[right] regular bands, is closed within the class of all semigroups satisfying the identity $ax = axa[xa = axa]$. In the present section, we shall show that a subclass of the class of all semigroups satisfying the identity $ax = axa[xa = axa]$ is saturated.

Theorem 3.3.1. Let U be a semigroup satisfying the identity $ax = axa$ and let $E(U)$, the set of all idempotents of U, be a semilattice. Then U is saturated.

In order to prove our result, we need the following useful remark which may be well known.

Remark 3.3.2. Let U be a semigroup satisfying the identity $ax = axa$. If $E(U)$ be the set of all idempotents of U, then $U^2 \subseteq E(U)$.

Proof. Suppose U be any semigroup satisfying the identity $ax = axa$. Suppose to the contrary that U is not saturated. Then, there exists a semigroup S containing U properly and such that $Dom(U,S) = S$. Take any $d \in Dom(U,S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m. As (1.4.1) is of minimal length, by Result 3.2.3, $x_i,\ y_i \in S \setminus U$ for all $i = 1, 2, \ldots, m$. Therefore, by Result 3.2.4,

$$x_i = z_i u_i,\ y_i = v_i t_i \tag{3.3.1}$$

for some $z_i,\ t_i \in S \setminus U;\ u_i,\ v_i \in U\ (i = 1, 2, \ldots, m)$. Now

$$\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 && \text{(by zigzag equations)} \\
&= (z_1 u_1) a_1 y_1 && \text{(by equalities (3.3.1) as } x_1 \in S \setminus U) \\
&= z_1 (u_1 a_1)(u_1 a_1) y_1 && \text{(by Remark 3.3.2)} \\
&= x_1 a_1 u_1 a_2 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 u_1 a_2 v_2 t_2 && \text{(by equalities (3.3.1) as } y_2 \in S \setminus U) \\
&= x_1 (a_1 u_1)(a_2 v_2)(a_2 v_2) t_2 && \text{(by Remark 3.3.2)} \\
&= x_1 (a_2 v_2)(a_1 u_1)(a_2 v_2) t_2 && \text{(by commutativity of } E(U)) \\
&= x_1 a_2 (v_2 a_1 u_1) a_2 y_2 && \text{(by zigzag equations)} \\
&= x_2 a_3 (w_2) a_2 y_2 && \text{(where } w_2 = v_2 w_1 a_2 \in U \text{ and } w_1 = a_1 u_1 \in U)
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&= x_{m-1}a_{2m-2}(w_{m-1})a_{2m-3}y_{m-1} && \text{(where } w_{m-1} = w_{m-2}w_{m-3}\cdots w_2w_1 \in U\text{)} \\
&= x_m a_{2m-1}(w_{m-1})a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= z_m u_m a_{2m-1}(w_{m-1}a_{2m-2})y_m && \text{(by equalities (3.3.1) as } x_m \in S \setminus U\text{)} \\
&= z_m(u_m a_{2m-1})(u_m a_{2m-1})(w_{m-1}a_{2m-2})y_m && \text{(by Remark 3.3.2)} \\
&= z_m(u_m a_{2m-1})(w_{m-1}a_{2m-2})(u_m a_{2m-1})y_m && \text{(by commutativity of } E(U)\text{)} \\
&= x_m(a_{2m-1}w_{m-1}a_{2m-2}u_m)a_{2m} && \text{(by zigzag equations)} \\
&\vdots \\
&= x_1a_1(w_m)a_{2m} (\text{where } w_m = w_{m-1}w_{m-2}w_{m-3}\cdots w_2w_1 \in U \\
& && \text{(by commutativity of } E(U)\text{)} \\
&= a_0(w_m)a_{2m} \in U && \text{(by zigzag equations)}
\end{aligned}
$$

$\Rightarrow d \in U$. Hence $Dom(U, S) \neq S$. □

Dually, we may prove the following:

Theorem 3.3.3. Let U be a semigroup satisfying the identity $xa = axa$ and $E(U)$, the set of all idempotents of U, be a semilattice. Then U is saturated. □

§ 3.4. QUASI-COMMUTATIVE SEMIGROUPS

Khan [58] has shown that a commutative semigroup satisfying a nontrivial identity of which at least one side has no repeated variable is saturated. We extend this result and show that a quasi-commutative semigroup satisfying a nontrivial identity of which at least one side has no repeated variable is saturated.

Definition 3.4.1[69]. An element a of a semigroup S is said to be *left[right] quasi-commutative* if for every element b in S, $ab = b^r a[ab = ba^r]$ holds for some positive integer $r \geq 1$.

Definition 3.4.2. A semigroup S is called *left[right] quasi-commutative* if every element of S is left[right] quasi-commutative. A semigroup S is called *quasi-commutative* if it both left and right quasi-commutative.

Proposition 3.4.3. Let S be a quasi-commutative semigroup. Then, for any elements $a, b \in S$ and positive integers s and $l(\geq 2)$

$$(ab^s)^l = a^{(1+r_1+r_2+\cdots+r_{l-1})}b^{ls} \; [(b^s a)^l = b^{ls}a^{(1+r_1+r_2+\cdots+r_{l-1})}] \tag{3.4.1}$$

for some positive integers $r_1, r_2, \ldots, r_{l-1}$.

Proof. We shall prove the equality (3.4.1) by induction on l. For $l = 2$, we have

$$\begin{aligned}
(ab^s)^2 &= ab^s ab^s \\
&= cab^s && (\text{take } c = ab^s) \\
&= a^{r_1}cb^s && (\text{by quasi-commutativity for some integer } r_1 \geq 1) \\
&= a^{r_1}(ab^s)b^s && (\text{as } c = ab^s) \\
&= a^{1+r_1}b^{2s}.
\end{aligned}$$

Therefore (3.4.1) holds for $l = 2$.

Suppose, inductively, that (3.4.1) holds for all positive integers k such that $2 \leq k \leq l-1$. Therefore, we have

$$(ab^s)^{l-1} = a^{(1+r_1+r_2+\cdots+r_{l-2})}b^{(l-1)s}. \tag{3.4.2}$$

Now

$$\begin{aligned}
(ab^s)^l &= (ab^s)^{l-1}ab^s \\
&= (a^{(1+r_1+r_2+\cdots+r_{l-2})}b^{(l-1)s})ab^s \qquad \text{(by equality (3.4.2))} \\
&= a^{(1+r_1+r_2+\cdots+r_{l-2})}(b^{(l-1)s}a)b^s \\
&= a^{(1+r_1+r_2+\cdots+r_{l-2})}(a^{r_{l-1}}b^{(l-1)s})b^s \\
&\qquad \text{(by quasi-commutativity for some integer } r_{l-1} \geq 1) \\
&= a^{(1+r_1+r_2+\cdots+r_{l-2}+r_{l-1})}b^{ls}
\end{aligned}$$

as required. □

By a nontrivial identity, we mean an identity not implied by quasi-commutativity and associativity combined.

Theorem 3.4.4. If a quasi-commutative semigroup U satisfies a nontrivial identity I such that one side of I has no repeated variable, then U is saturated.

Proof. Without loss of generality we shall assume that I has the following form:

$$x_1x_2\cdots x_n = w(x_1, x_2, \ldots x_n, x_{n+1}, \ldots, x_{n+m}) \tag{3.4.3}$$

where $w(x_1, x_2, \ldots x_n, x_{n+1}, \ldots, x_{n+m}) = {x_1}^{l_1}\, {x_2}^{l_2} \cdots\, {x_{n+m}}^{l_{n+m}}$, l_i and l_{n+k} are integers such that $l_i \geq 0$, $l_{n+k} \geq 1$ for $i = 1, 2, \cdots, n(\geq 1)$ and $k = 1, 2, \cdots, m(\geq 0)$.

Now in order to prove the theorem, it is sufficient to prove it in the case when $l_1 \geq 2$. For, if $m \geq 1$, then by substituting x_1^2 for x_{n+m} and using quasi-commutativity, we obtain a new identity implied by (3.4.3), namely

$$x_1x_2\cdots x_n = {x_1}^{l_1+r(2l_{n+m})}\, {x_2}^{l_2} \cdots\, {x_{n+m-1}}^{l_{n+m-1}}$$

for some positive integer r. If $m = 0$, since the identity (3.4.3) is nontrivial, by using quasi-commutativity, we can assume that $l_1 \geq 2$. In either case we have that U satisfies an identity of the form (3.4.3) with $l_1 \geq 2$.

Let us now assume to the contrary that U is not saturated. Therefore, there exists a semigroup S containing U properly and such that $Dom(U, S) = S$.

Lemma 3.4.5. For all $a \in U$, $x \in S \setminus U$, and $y \in S$, there exists $w \in U$ such that $xay = xa^2wy$.

Proof. Since $x \in S \setminus U$, by Result 3.2.4, we have

$$x = zc_1c_2\cdots c_n \tag{3.4.4}$$

for some $z \in S \setminus U$ and $c_1, c_2, \cdots, c_n \in U$.

Then

$$\begin{aligned}
xay &= zc_1c_2\cdots c_nay \\
&= z(a^tc_1)c_2\cdots c_ny && \text{(by quasi-commutativity for some integer } t \geq 1) \\
&= zw(a^tc_1, c_2, \ldots, c_n, c_{n+1}, \ldots, c_{n+m})y && \text{(for any } c_{n+1}, \ldots, c_{n+m} \in U, \\
& && \text{as } U \text{ satisfies (3.4.3))} \\
&= z((a^tc_1)^{l_1}c_2^{l_2}\cdots c_n^{l_n}c_{n+1}^{l_{n+1}}\cdots c_{n+m}^{l_{n+m}})y \\
&= z(a^{l_1t}c_1^{(1+r_1+r_2+\cdots+r_{l_1-1})}c_2^{l_2}\cdots c_n^{l_n}c_{n+1}^{l_{n+1}}\cdots c_{n+m}^{l_{n+m}})y \\
& \qquad \text{(by Proposition 3.4.3 for some integers } r_1, r_2, \ldots, r_{l_1-1} \geq 1) \\
&= z(a^{l_1t}c_1^{(1+r_1+r_2+\cdots+r_{l_1-1})-l_1})c_1^{l_1}c_2^{l_2}\cdots c_n^{l_n}c_{n+1}^{l_{n+1}}\cdots c_{n+m}^{l_{n+m}}y \\
&= z(a^{l_1t}c_1^{p})c_1^{l_1}c_2^{l_2}\cdots c_n^{l_n}c_{n+1}^{l_{n+1}}\cdots c_{n+m}^{l_{n+m}}y && \text{(where } p = 1 + r_1 + r_2 + \cdots + r_{l_1-1} \geq 2)
\end{aligned}$$

$$= zc_1^{l_1}c_2^{l_2}\cdots c_n^{l_n}c_{n+1}^{l_{n+1}}\cdots c_{n+m}^{l_{n+m}}(a^{l_1t}c_1^p)^r y \qquad \text{(by quasi-commutativity)}$$

$$= zw(c_1, c_2, \ldots, c_{n+m})(a^{l_1t}c_1^p)^r y$$

$$= zc_1c_2\cdots c_n(a^{l_1t}c_1^p)^r y$$

$$= zc_1c_2\cdots c_n(bc_1^p)^r y \qquad \text{(where } b = a^{l_1t}\text{)}$$

$$= zc_1c_2\cdots c_n(b^{(1+q_1+q_2+\cdots+q_{l_r-1})}c_1^{pr})y$$

(by Proposition 3.4.3, as $q_1, q_2, \ldots, q_{l_r-1} \geq 1$)

$$= zc_1c_2\cdots c_n(b^u c_1^{pr})y \qquad \text{(where } u = 1 + q_1 + q_2 + \cdots + q_{l_r-1}\text{)}$$

$$= zc_1c_2\cdots c_n(a^{l_1tu}c_1^{pr})y \qquad \text{(as } b = a^{l_1t}\text{)}$$

$$= zc_1c_2\cdots c_n a^2(a^{l_1tu-2}c_1^{pr})y$$

$$= xa^2wy \qquad \text{(by equation (3.4.4) and where } w = a^{l_1tu-2}c_1^{pr} \in U\text{)}$$

This proves the lemma.

Now to complete the proof of the theorem, we take any $d \in S \setminus U$, and let (1.4.1) be a zigzag for d over U of minimal length m. Then

$$d = x_1a_1y_1$$

$$= x_1a_1^2w_1y_1 \qquad \text{(by Lemma 3.4.5 as } w_1 \in U\text{)}$$

$$= x_1a_1(a_1w_1)y_1$$

$$
\begin{aligned}
&= x_1 a_1 (w_1^{r_1} a_1) y_1 && \text{(by left quasicommutativity of } U \\
& && \text{and for some integer } r_1 \geq 1) \\
&= (x_1 a_1) w_1^{r_1} (a_1 y_1) \\
&= x_1 (a_1 w_1^{r_1})(a_2 y_2) && \text{(by zigzag equations)} \\
&= x_1 a_2 (a_1 w_1^{r_1})^{r_2} y_2 && \text{(by right quasicommutativity of } U \\
& && \text{and for some integer } r_2 \geq 1) \\
&= x_2 a_3 ((a_1 w_1^{r_1})^{r_2} y_2) && \text{(by zigzag equations)} \\
&= x_2 a_3^2 w_3 ((a_1 w_1^{r_1})^{r_2} y_2) && \text{(by Lemma 3.4.5 as } w_3 \in U) \\
&= x_2 a_3 a_3 (w_3 (a_1 w_1^{r_1})^{r_2}) y_2 \\
&= (x_2 a_3)(w_3 (a_1 w_1^{r_1})^{r_2})^{r_3} (a_3 y_2) && \text{(by left quasicommutativity of } U \\
& && \text{and for some integer } r_3 \geq 1) \\
&= (x_2 a_3) w_3^{r_3} a_1^{r_2 r_3} w_1^{r_1 r_2 r_3} (a_3 y_2) \\
&= (x_2 a_3) w_3^{r_3} a_1^{t_1} w_1^{z_1} (a_3 y_2) && \text{(where } z_1 = r_1 r_2 r_3 \text{ and } t_1 = r_2 r_3) \\
&= (x_2 a_3) a_1^{t_1} w_3^{r r_3} w_1^{z_1} (a_3 y_2) && \text{(by right quasicommutativity of } U \\
& && \text{and for some integer } r \geq 1) \\
&= (x_2 a_3) a_1^{t_1} w_3^{z_3} w_1^{z_1} (a_3 y_2) && \text{(where } z_3 = r r_3) \\
&\vdots \\
&= (x_m a_{2m-1})(a_{2m-3}^{t_{2m-3}} a_{2m-5}^{t_{2m-5}} \cdots a_3^{t_3} a_1^{t_1})(w_{2m-1}^{z_{2m-1}} w_{2m-3}^{z_{2m-3}} \cdots w_3^{z_3} w_1^{z_1})(a_{2m-1} y_m)
\end{aligned}
$$

$$= x_{m-1}(a_{2m-2}a_{2m-3}^{t_{2m-3}})(a_{2m-5}^{t_{2m-5}} \cdots a_3^{t_3} a_1^{t_1})(w_{2m-1}^{z_{2m-1}} w_{2m-3}^{z_{2m-3}} \cdots w_3^{z_3} w_1^{z_1})(a_{2m-1}y_m)$$

(by zigzag equations)

$$= x_{m-1}(a_{2m-3}^{t_{2m-3}} a_{2m-2}^{p_{2m-2}})(a_{2m-5}^{t_{2m-5}} \cdots a_3^{t_3} a_1^{t_1})(w_{2m-1}^{z_{2m-1}} w_{2m-3}^{z_{2m-3}} \cdots w_3^{z_3} w_1^{z_1})(a_{2m-1}y_m)$$

(by right quasicommutativity of U

and for some integer $p_{2m-2} \geq 1$)

$$= x_{m-1}a_{2m-3}a_{2m-3}^{(t_{2m-3}-1)} a_{2m-2}^{p_{2m-2}}(a_{2m-5}^{t_{2m-5}} \cdots a_3^{t_3} a_1^{t_1})(w_{2m-1}^{z_{2m-1}} w_{2m-3}^{z_{2m-3}} \cdots w_3^{z_3} w_1^{z_1})$$
$$(a_{2m-1}y_m)$$

$$= x_{m-2}a_{2m-4}a_{2m-3}^{q_{2m-3}} a_{2m-2}^{p_{2m-2}}(a_{2m-5}^{t_{2m-5}} \cdots a_3^{t_3} a_1^{t_1})(w_{2m-1}^{z_{2m-1}} w_{2m-3}^{z_{2m-3}} \cdots w_3^{z_3} w_1^{z_1})(a_{2m-1}y_m)$$

(where $q_{2m-3} = t_{2m-3} - 1$ and by zigzag equations)

$\vdots$

$$= x_1 a_1 a_2^{p_2} a_3^{q_3} a_4^{p_4} \cdots a_{2m-4}^{p_{2m-4}} a_{2m-3}^{q_{2m-3}} a_{2m-2}^{p_{2m-2}} w'(a_{2m-1}y_m)$$

(where $w' = w_{2m-1}^{z_{2m-1}} w_{2m-3}^{z_{2m-3}} \cdots w_3^{z_3} w_1^{z_1}$)

$$= a_0 a_2^{p_2} a_3^{q_3} a_4^{p_4} \cdots a_{2m-4}^{p_{2m-4}} a_{2m-3}^{q_{2m-3}} a_{2m-2}^{p_{2m-2}} w'(a_{2m})$$

(where $w' = w_{2m-1}^{z_{2m-1}} w_{2m-3}^{z_{2m-3}} \cdots w_3^{z_3} w_1^{z_1}$ and by zigzag equations)

$$= a_0 a_2^{p_2} a_3^{q_3} a_4^{p_4} \cdots a_{2m-4}^{p_{2m-4}} a_{2m-3}^{q_{2m-3}} a_{2m-2}^{p_{2m-2}} a_{2m}(w'^m)$$

$$= a_0 a_2^{p_2} a_3^{q_3} a_4^{p_4} \cdots a_{2m-4}^{p_{2m-4}} a_{2m-3}^{q_{2m-3}} a_{2m-2}^{p_{2m-2}} a_{2m}(w) \in U \qquad (\text{where } w = w'^m \in U)$$

a contradiction, as required. □

CHAPTER 4

ON SUPERSATURATED SEMIGROUPS

§ 4.1. INTRODUCTION

The class of supersaturated semigroups has not received much attention before as there was no known example of a saturated semigroup with a morphic image not saturated. Indeed, many of the known classes of saturated semigroups are closed under the taking of morphic images. In [37], Higgins has constructed an example of a semigroup, which by itself, is saturated, but whose morphic image is not saturated (Higgins [31]). In this chapter, we discuss supersaturated semigroups and ideals.

§ 4.2. SUPERSATURATED SEMIGROUP AND AMALGAMS

In this section, after defining a supersaturated semigroup, we present an example due to Higgins [37], which shows that the morphic image of a saturated semigroup need not be saturated. Further, we give a brief exposition of semigroup amalgams and their relationship with dominions, which we shall be using to prove the main result of the next section.

Definition 4.2.1. A semigroup S is said to be *supersaturated* if every morphic image of S is saturated.

We note that every epimorphism from a semigroup S is *onto* if and only if S is supersaturated.

The following example, from Higgins [37], shows that the morphic image of a saturated semigroup need not be saturated in general. In fact, this is the first known example of a semigroup which is not supersaturated.

Example 4.2.2. Let U be the relatively free semigroup in the variety $[xy = yx,\ x^2 = 0]$

on the denumerable generating set $X = \{x_1, x_2, \ldots\}$. In [36], Higgins has shown that there exists a countable member of this variety (a morphic image of U) that is not saturated. We now prove that U is saturated. First observe that U can be regarded as consisting of non-empty finite subsets of X with multiplication defined by

$$ab = \begin{cases} a \cup b & \text{if } a \cap b \neq \phi \\ \\ 0 & \text{otherwise,} \end{cases}$$

where a, b are non-empty finite subsets of X, and 0 is an additional symbol representing the zero of U.

Now suppose that U is properly epimorphically embedded in a semigroup S. Let $d \in S \setminus U$, and let Z be a zigzag in S over U with value d and minimum length m. Consider the first zigzag equality $a_0 = y_1 a_1$. Since Z has a minimum length, $y_1 \in S \setminus U$ and so, by the Zigzag Theorem, we may successively factorize y_1 as

$y_1 = y_1^{(1)} u_1 = y_1^{(2)} u_1 u_2 = \cdots = y_1^{(i)} u_1 u_2 \cdots u_i = \cdots$ where $u_j \in U$, $y_1^{(j)} \in S \setminus U$ for $j = 1,\ 2, \ldots$.

Then $u_1 u_2 \ldots u_i a_1 \neq 0$, $\forall\ i = 1, 2, \ldots$ as otherwise we would have $d = y_1^{(i)} 0 t_1 = 0$ and $0 \in U$, a contradiction, as from [57], the zero of U is also the zero of $Dom(U, S)$.

We thus have $a_1 \subset u_1 a_1 \subset \ldots \subset u_1 u_2 \cdots u_i a_1 \subset \cdots$ for some i, and therefore, we have $u_1 u_2 \cdots u_i a_1 \not\subseteq a_0$. Choose $t_k \in u_1 u_2 \cdots u_i a_1 \setminus a_0$. Then $a_0 t_k \neq 0$. But on the other hand $a_0 t_k = y_1 a_1 t_k = y_1^{(i)} u_1 u_2 \cdots u_i a_1 t_1 = y_1^{(i)} 0 = 0$, a contradiction. Therefore U is saturated.

The systematic study of semigroup amalgams was initiated by Howie ([44]- [50]). Preston [75], Hall [23, 25, 27] and Imaoka [55, 56] developed a representational approach. Renshaw showed [78, 79] how this approach, expressed in homological terms, could be seen as similar to Cohn's [11] work on amalgamation theory for rings.

In this section, we shall only consider amalgams of the semigroups S and T with a common core subsemigroup U denoted by $[S, T; U]$. The amalgam $[S, T; U]$ is, then, a partial semigroup: some products are meaningful and $(xy)z = x(yz)$ provided both sides are defined. A natural question is, whether or not the amalgam $[S, T; U]$ can be embedded in another semigroup W (i.e., $[S, T; U] \subseteq W$ and previously defined products in the amalgam are unaltered).

In 1927, Schreier showed that any group amalgam is always embeddable in a group, but this simple answer does not suffice for semigroups. A natural candidate for a semigroup W into which the amalgam $[S, T; U]$ might be embedded is the so-called *amalgamated free product* of S and T over U, which is constructed via the free product of S and T

Definition 4.2.3. The *free product* of two semigroups S and T, denoted by $S * T$, is defined to be consisting of finite sequences or words whose letters alternately come from S and T. The product of two members w_1, w_2 of $S * T$ is defined by concatenation if the last letter of w_1 and the first letter of w_2 do not come from the same semigroup, otherwise $w_1 w_2$ is defined by first forming the concatenated word, and then by performing the multiplication of the last letter of w_1 and first letter of w_2 in the semigroup (S or T) of which both of them are members.

The above definition of free product can be generalized to any arbitrary family $\{S_i : i \in I\}$ of semigroups. We denote the free product of the family $\{S_i : i \in I\}$ of semigroups by $\Pi^* S_i$.

Definition 4.2.4. The *amalgamated free product* of the amalgam $\mathcal{U}$ is defined as the quotient semigroup of the ordinary free product $\Pi^* S_i$ in which for each i and j in I, the image $u\phi_i$ of an element u of U in S_i is identified with its image $u\phi_j$in S_j, and we write it as $P = \Pi^*_U S_i$. More precisely, if we denote by θ_i, the natural monomorphism from S_i into $\Pi^* S_i$, then we define $P = \Pi^*_U S_i$ to be $(\Pi^* S_i/\rho)$, where ρ is the congruence on $\Pi^* S_i$ generated by the subset

$$R = \{(u\phi_i\theta_i, u\phi_j\theta_j) : u \in U, i, j \in I\},$$

of $\Pi^* S_i \times \Pi^* S_i$.

Result 4.2.5. ([51, Chapter VII, Theorem 1.11]). The semigroup amalgam

$$\mathcal{U} = [\{S_i : i \in I\};\ U\ ; \{\phi_i : i \in I\}]$$

is embeddable in a semigroup if and only if it is naturally embedded in its free product.

§ 4.3. SUPERSATURATED SEMIGROUPS AND IDEALS

In [36], Higgins showed that, a semigroup U is saturated [supersaturated] if the ideal U^n is saturated [supersaturated]. Whether or not the converse of the above result is true, is an open question. But Higgins [37], had shown that the converse holds in some cases and proved that if S is a supersaturated semigroup, then any commutative globally idempotent ideal of S is also supersaturated. Khan and Shah [65] generalized this result by taking U as a permutative globally idempotent ideal satisfying a permutation identity $x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}$ for which $i_1 = 1$ and $i_n \neq n$ and thus, relaxed the commutativity of U. In this section, we extend this result by taking U as a permutative globally idempotent ideal satisfying a seminormal permutation identity and, thus, relax the right semicommutativity of U. Further, we enlarge the class of supersaturated globally idempotent ideals of a supersaturated semigroup by showing that a globally idempotent ideal of a supersaturated semigroup satisfying the identity $axa = ax[axa = xa]$ is supersaturated.

Definition 4.3.1. A semigroup S is said to be right reductive with respect to X if $xa = xb$ for all x in $X \Rightarrow a = b$ $(a, b \in S)$, where X is a subset of S. Dually we say that a semigroup S is left reductive with respect to X if $ax = bx$ $\forall x \in X \Rightarrow a = b$ $(a, b \in S)$. The following result is from Higgins [36].

Result 4.3.2 ([36, Theorem 8]). A semigroup U is saturated [supersaturated] if the ideal U^n is saturated [supersaturated] (for some natural number n). In particular, a finite semigroup is saturated [supersaturated] if the ideal generated by the idempotents is saturated [supersaturated].

This result, for saturated commutative semigroups, first appeared in [57]. However, Result 4.3.2 was not originally stated for supersaturated semigroups, but the proof goes through with essentially no alteration under this hypothesis.

Whether or not the converse of the above result is true is still an open question. Since $\Im_X$, the semigroup of all transformations on the set X, is absolutely closed, subsemigroups of absolutely closed or saturated semigroups need not be absolutely closed or saturated, in general. But it is not known whether or not an ideal of a saturated [absolutely closed] semigroup is saturated [absolutely closed]. In [37], Higgins, however, has shown that the converse of the above result holds in some special cases and has proved that if S is supersaturated commutative semigroup, then the same is true for any globally idempotent ideal. He has, in fact, shown the following:

Result 4.3.3 ([37, Theorem 14]). Let S be a saturated semigroup and suppose that U is a commutative ideal of S such that U^n is globally idempotent for some natural number n. Then U is supersaturated.

The next result from Khan [61], will be needed to complete the proof of main theorem of this section.

Result 4.3.4 ([61, Theorem 6.4]). Let Ω be the class of all globally idempotent semigroups and let

$$x_1x_2...x_n = x_{i_1}x_{i_2}...x_{i_n} \tag{4.3.1}$$

be any nontrivial permutation identity. Then (4.3.1) is equivalent with respect to Ω to
(i) commutativity if $i_1 \neq 1$ and $i_1 \neq n$;
(ii) left normality if $i_1 = 1$ and $i_n \neq n$;
(iii) right normality if $i_1 \neq 1$ and $i_n = n$;
(iv) normality if $i_1 = 1$ and $i_n = n$.

Theorem 4.3.5. Let S be a supersaturated semigroup and let U be any ideal of S satisfying a seminormal permutation identity. If U^n is globally idempotent for some natural number n, then U is supersaturated.

First we prove a lemma which is very crucial for the proof of the above theorem.

Lemma 4.3.6. Suppose that a globally idempotent semigroup U is not supersaturated. Then there exists a non-surjective epimorphism $\phi : U \to V$ such that V is right and left reductive with respect to $U\phi$.

Proof. As U is not supersaturated, let $\alpha : U \to V$ be a non-surjective epimorphism. Define a relation ρ on V as $s\rho t$ if $asu = atu$ $\forall a, u \in U\alpha$. Clearly ρ is an equivalence relation. Now take any $d \in V \setminus U\alpha$ and $s\rho t$. Then $d = xa$ for some $x \in V \setminus U\alpha$ and $a \in U\alpha$. Now $dsu = (xa)su = x(asu) = x(atu) = (xa)tu = dtu$. Hence, for any $w \in V$, a, u in $U\alpha$, $wsu = wtu \Rightarrow awsu = awtu \Rightarrow ws\rho wt$. So, ρ is a left congruence. Similarly, we take $d = uy$. Therefore $asd = as(uy) = (asu)y = (atu)y = at(uy) = atd$. Now for any $w \in V$, $a, u \in U\alpha$, $asw = atw \Rightarrow aswu = atwu \Rightarrow sw\rho tw$. Therefore ρ is a right congruence. Hence ρ is a congruence.

We denote the images of $U\alpha$ and V under $\rho^\natural$ by $\overline{U}$ and $\overline{V}$respectively and shall similarly abbreviate $u\rho$, $v\rho$ by $\overline{u}$, $\overline{v}$ $(u \in U,\, v \in V)$.

From the commutative diagram

$$\begin{tikzcd} U\alpha \arrow[r, "i", "(epi)"'] \arrow[d, "\rho^\natural|U\alpha"'] & V \arrow[d, "\rho^\natural"] \\ \bar{U} \arrow[r, "\bar{i}"] & \bar{V} \end{tikzcd}$$

one may easily see that the inclusion map $\bar{i} : \overline{U} \to \overline{V}$ is epi.

Now, we show that $\overline{V}$ is right and left reductive with respect to $\overline{U}$.

$\overline{V}$ **is right reductive.** Let $\overline{v_1} \neq \overline{v_2}$. Then $v_1\rho \neq v_2\rho \Rightarrow (v_1, v_2) \notin \rho \Rightarrow xv_1y \neq xv_2y$ for some $x, y \in U\alpha$. Since U is globally idempotent, so is $U\alpha$. Therefore $x = x_1x_2$ for some $x_1, x_2 \in U\alpha$. Then we have $x_2v_1y \neq x_2v_2y$. For if $x_2v_1y = x_2v_2y$, then $xv_1y = xv_2y$. This is a contradiction as $xv_1y \neq xv_2y \Rightarrow x_1(x_2v_1)y \neq x_1(x_2v_2)y \Rightarrow \overline{x_2v_1} \neq \overline{x_2v_2} \Rightarrow \overline{x}_2\overline{v}_1 \neq \overline{x}_2\overline{v}_2$. Therefore, $\overline{V}$ is right reductive with respect to $\overline{U}$.

$\overline{V}$ **is left reductive.** Let $\overline{v_1} \neq \overline{v_2}$. Then $v_1\rho \neq v_2\rho \Rightarrow (v_1, v_2) \notin \rho \Rightarrow xv_1y \neq xv_2y$ for some $x, y \in U\alpha$. Since U is globally idempotent, so is $U\alpha$. Therefore $y = y_1y_2$ for some $y_1, y_2 \in U\alpha$. Then, we have $xv_1y_1 \neq xv_2y_1$. For if $xv_1y_1 = xv_2y_1$, then $xv_1y = xv_2y$. This is a contradiction as $xv_1y \neq xv_2y \Rightarrow$ $x(v_1y_1)y_2 \neq x(v_2y_1)y_2 \Rightarrow \overline{v_1y_1} \neq \overline{v_2y_1} \Rightarrow \overline{v}_1\overline{y}_1 \neq \overline{v}_2\overline{y}_1$. Hence, $\overline{V}$ is left reductive with respect to $\overline{U}$.

Finally we prove that the inclusion map $\bar{i} : \overline{U} \to \overline{V}$ is not surjective. Let $d \in V \setminus U\alpha$ and $s\rho t$. Then $d = uy$ for some $u \in U\alpha$, $y \in V \setminus U\alpha$. Let $a, v \in U\alpha$. Then $uav \neq uyv$ (as $uav \in U\alpha$). For if $uav = uyv \;\Rightarrow\; ua = uy$ (as $\overline{V}$ is right reductive with respect to $\overline{U}$). As $ua \in U\alpha$ and $d = uy \in V \setminus U\alpha$, we get a contradiction. Hence $uav \neq uyv$ implies that $(a, y) \notin \rho$ which implies that $\bar{a} \neq \bar{y} \;\; \forall a \in U\alpha$. Therefore $\bar{i}(\bar{a}) = \bar{a} \neq \bar{y}$ for some $\bar{y} \in \overline{V}$ such that $\bar{y} \notin \overline{U}$. So $\bar{i}$ is not surjective.

Therefore $\phi = \alpha\rho^{\natural} : U \to \overline{V}$ is a non-surjective epimorphism from U in to $\overline{V}$ which is right reductive with respect to the image $U\phi$ of U. Hence the lemma is proved. $\Box$

Proof of the main Theorem. If we prove that U^n is supersaturated, then the theorem follows by Result 4.3.3. So, we assume that U is globally idempotent ideal satisfying a seminormal permutation identity. Suppose to the contrary that U were not supersaturated. Then, by Lemma 4.3.6, there exists a non-surjective epimorphism $\phi : U \longrightarrow \overline{V}$ such that $\overline{V}$ is right and left reductive with respect to $U\phi$ (which we shall denote by $\overline{U}$ up to isomorphism).

Let $\rho = \phi \circ \phi^{-1} \cup I_s$. Then, clearly ρ is an equivalence relation on S. Next, we show that ρ is a congruence on S. For this we are required to show that if $u, v \in U$ and $w \in S \setminus U$, then $u\phi = v\phi$ implies that $(uw)\phi = (vw)\phi$ and $(wu)\phi = (wv)\phi$.

To prove the first equality, suppose that $u, v \in U$ and $w \in S \setminus U$ and $(uw)\phi = (vw)\phi$. Since $\overline{V}$ is right reductive with respect to $\overline{U}$, there exist $x \in U$ such that $x\phi(uw)\phi \neq x\phi(vw)\phi$, then $(x(uw))\phi \neq (x(vw))\phi$. Since $\overline{V}$ is left reductive with respect to $\overline{U}$, there exists $y \in U$ such that $(x(uw))\phi y\phi \neq (x(vw))\phi y\phi$. Then $((x(uw))y)\phi \neq ((x(vw))y)\phi$. Since U is globally idempotent, $x = x_1x_2$ for some $x_1, x_2 \in U$ and $y = y_1y_2$ for some $y_1, y_2 \in U$. Thus, we have

$$(x_1x_2(uw)y_1y_2)\phi \neq (x_1x_2(vw)y_1y_2)\phi$$

$$\Rightarrow (x_1x_2u(wy_1)y_2)\phi \neq (x_1x_2v(wy_1)y_2)\phi.$$

Now, by Result 4.3.4, we have

$$(x_1x_2(wy_1)uy_2)\phi \neq (x_1x_2(wy_1)vy_2)\phi$$

$$\Rightarrow ((x_1x_2w)y_1uy_2)\phi \neq ((x_1x_2w)y_1vy_2)\phi.$$

Again, by Result 4.3.4, we get

$$((x_1x_2w)uy_1y_2)\phi \neq ((x_1x_2w)vy_1y_2)\phi$$

$$\Rightarrow (x_1x_2(wu)y_1y_2)\phi \neq (x_1x_2(wv)y_1y_2)\phi$$

$$\Rightarrow (x(wu)y)\phi \neq (x(wv)y)\phi$$

$$\Rightarrow (x(wu))\phi y\phi \neq (x(wv))\phi y\phi$$

$$\Rightarrow (x(wu))\phi \neq (x(wv))\phi$$

$$\Rightarrow ((xw)u))\phi \neq ((xw)v))\phi$$

$$\Rightarrow (xw)\phi u\phi \neq (xw)\phi v\phi \Rightarrow u\phi \neq v\phi.$$

Therefore, the statement

$u\phi = v\phi \Rightarrow (uw)\phi = (vw)\phi \Rightarrow \rho$ is a right congruence.

Next we show that ρ is a left congruence. Suppose that $u, v \in U$ and $w \in S \setminus U$ and $(uw)\phi \neq (vw)\phi$. Since $\overline{V}$ is right reductive with respect to $\overline{U}$, there exists $x \in U$ such that $x\phi(wu)\phi \neq x\phi(wv)\phi$. Now $(x(wu))\phi \neq (x(wv))\phi \Rightarrow ((xw)u)\phi \neq ((xw)v))\phi \Rightarrow (xw)\phi u\phi \neq (xw)\phi v\phi$. Hence $u\phi \neq v\phi$.

Again we conclude that the statement $u\phi = v\phi$ implies that $(wu)\phi = (wv)\phi$. Therefore ρ is a left congruence and, hence, a congruence. Denote S/ρ by $\overline{S}$. Then $U^\natural = \overline{U}$ (up to isomorphism).

Now we form the amalgam A of $\overline{S}$ and $\overline{V}$ with core $\overline{U}$. We extend the partial operation on A to an associative multiplication. For this take any $a \in \overline{S} \setminus \overline{U} (= S \setminus U)$, $v \in \overline{V} \setminus \overline{U}$ and factorize v as $v = u_1y_1 = y_1'u_1'$ (where $u_1, u_1' \in \overline{U}$; $y_1, y_1' \in \overline{V} \setminus \overline{U}$). Now define $av = (au_1)y_1$ and $va = y_1'(u_1'a)$. We first show that this is a well defined binary operation. For this suppose that $v = u_2y_2 = y_2'u_2'$ (where $u_2, u_2' \in U$ and $y_2, y_2' \in \overline{V} \setminus \overline{U}$). Then for any $x \in \overline{U}$, as $u_1y_1 = u_2y_2$, we have

$(xa)u_1y_1 = (xa)u_2y_2$
$\Rightarrow ((xa)u_1)y_1 = ((xa)u_2)y_2$ (by associativity of $\overline{V}$)
$\Rightarrow (x(au_1))y_1 = (x(au_2))y_2$ (by associativity of $\overline{S}$)

$\Rightarrow x((au_1)y_1) = x((au_2)y_2)$ (by associativity of $\overline{V}$).

As $\overline{V}$ is right reductive with respect to $\bar{U}$, we have that $(au_1)y_1 = (au_2)y_2$ and, therefore, the operation is well defined.

Now, we verify the associativity of the above operation. For any $a' \in S \setminus U$, we have $a'(av) = a'(au_1y_1) = (a'au_1)y_1 = (a'a)u_1y_1 = (a'a)v$ (by associativity of $\overline{V}$ and $\overline{S}$ respectively).

Similarly, for any $v' \in V$, it can be shown that $(av)v' = a(vv')$. The only case that requires some attention is to show that $(av)a' = a(va')$, where $a' \in S, v \in \overline{V}$. For this, factorize v as $v = a_1ya_2$ (where $a_1, a_2 \in \overline{V} \setminus \overline{U}$). Now

$$\begin{aligned}
(av)a' &= (a(a_1ya_2))a' \quad (\text{as } v = a_1ya_2) \\
&= ((aa_1)ya_2)a' \quad (\text{as } aa_1 \in \overline{U} \subseteq \overline{S}) \\
&= (aa_1)(ya_2)a' \quad (\text{by associativity of } \overline{V}) \\
&= (aa_1)y(a_2a') \quad (\text{as } a_2a' \in \overline{U} \subseteq \overline{S}) \\
&= (aa_1)(ya_2a') \quad (\text{by associativity of } \overline{V}) \\
&= a(a_1ya_2a') \quad (\text{by associativity of } \overline{V}) \\
&= a(va'),
\end{aligned}$$

as required.

We, now, have $\overline{S} \neq A = Dom(\overline{S}, A)$.

This contradicts the fact that S is supersaturated.

Hence the theorem is proved. $\square$

Now, we further, enlarge the class of supersaturated globally idempotent ideals of a supersaturated semigroup by showing that a globally idempotent ideal of a supersaturated semigroup satisfying the identity $axa = ax[axa = xa]$ is supersaturated.

The following result is very crucial for the proof of our next theorem.

Result 4.3.7. ([65, Lemma 2.5]). Suppose that a globally idempotent semigroup U is not supersaturated. Then there exists a non-surjective epimorphism $\phi : U \to V$ such that V is right reductive with respect to $U\phi$.

Theorem 4.3.8. Let S be a supersaturated semigroup and let U be any ideal of S satisfying the identity $axa = ax$. If U^n is globally idempotent for some natural number n, then U is supersaturated.

Proof: By Result 4.3.3, the theorem is proved if we prove that U^n is supersaturated. Therefore, without loss of generality, we may assume that U be a globally idempotent ideal satisfying the identity $axa = ax$. Let us suppose to contrary that U were not supersaturated. Then, by Result 4.3.7, there exists a non-surjective epimorphism $\phi : U \to \overline{V}$ such that $\overline{V}$ is right reductive with respect to $U\phi$, denoted by $\overline{U}$ (upto isomorphism). Let $\rho = \phi o \phi^{-1} \cup 1_s$. It is clear that ρ is an equivalence relation on S. Next we show that ρ is a congruence on S. For this we require to show that if $u, v \in U$ and $w \in S \backslash U$, then $u\phi = v\phi$ implies that $(uw)\phi = (vw)\phi$ and $(wu)\phi = (wv)\phi$. We will prove only the first equality as rest of the proof including that of the latter equality follows exactly on the same lines as in the case of the proof of [65, Theorem 2.6]

Suppose that $u, v \in U$, $w \in S \backslash U$ and $(uw)\phi \neq (vw)\phi$. Since $\overline{V}$ is right reductive with respect to $\overline{U}$, there exists $x \in U$ such that $x\phi(uw)\phi \neq x\phi(vw)\phi$ Then $(x(uw))\phi \neq (x(vw))\phi$ which implies that $((xu)w)\phi \neq ((xv)w)\phi$. Since U satisfies identity $axa = ax$. Therefore we have $((xux)w)\phi \neq ((xvx)w)\phi \Rightarrow ((xu)xw)\phi \neq ((xv)xw)\phi$ $\Rightarrow (xu)\phi(xw)\phi \neq (xv)\phi(xw)\phi$. which in turn implies that $(xu)\phi \neq (xv)\phi$ again, it implies that $x\phi u\phi \neq x\phi v\phi$ which in turn implies that $u\phi \neq v\phi$. Therefore the statement $u\phi = v\phi$ implies that $(uw)\phi = (vw)\phi$. Hence ρ is a right congruence.

Dually, we may prove the following:

Theorem 4.3.9. Let S be a supersaturated semigroup and let U be any ideal of S satisfying the identity $axa = xa$. If U^n is globally idempotent for some natural number n, then U is supersaturated. $\square$

CHAPTER 5

ON ZIGZAG THEOREM AND ON ABSOLUTELY CLOSED SEMIGROUPS

§ 5.1. INTRODUCTION

The aim of this chapter is to establish the theorems on Isbell's Zigzag Theorem for commutative semigroups and on absolutely closed semigroups. In [54], Howie and Isbell extended Isbell's Zigzag Theorem, by using free products of commutative semigroups, for the category of all commutative semigroups. Stenstrom [85] provided, by using tensor product of monoids, a new proof of the celebrated Isbell's Zigzag Theorem in the category of all semigroups. In Section 5.2, based on Stenstrom's approach, we provide a new and short algebraic proof of the Howie and Isbell's result [54, Theorem 1.1] for the category of all commutative semigroups. In the next Section 5.3, we show that a subclass of the class of all regular medial semigroups and the class of all globally idempotent commutative semigroups satisfying the identity $ax = axa[xa = axa]$ are absolutely closed.

§ 5.2. ZIGZAG THEOREM FOR COMMUTATIVE SEMIGROUP

In 1965, Isbell introduced celebrated Zigzag Theorem in the category of semigroups. His proof was topological in nature. Later on, several author's proved this theorem by different techniques. In 1974, J.M. Philip [73] gave his geometric argument to improve Isbell's result. In [85] and [86], Stenstrom, H.H. Storrer provided an algebraic proof using tensor products. The proof also appears in J.M. Howie's book [51]. P. M. Higgins also gave a short proof of Isbells Zigzag Theorem (see [41]). Both Storrers and Higgins proofs establish the theorem for monoids, and, then derive it for semigroups.

The Zigzag Theorem also holds in the category of commutative semigroups (that is, if A is a commutative semigroup and B a subsemigroup of A, then the dominion of

B in A with respect to the category of all semigroups is equal to the dominion of B in A with respect to the category of all commutative semigroups). This was established by Isbell and Howie shortly after Isbells first paper on dominions [57]. The proof in this instance is completely algebraic, and relies on the free sum of two commutative semigroups.

In this Section, we prove Isbell's Zigzag Theorem for the category of commutative semigroups. This algebraic proof is new and short, is based on Stenstrom's approach, and quite different from the Howie and Isbell's result [54, Theorem 1.1].

The following important results shall be used to prove our main Theorem.

Result 5.2.1([53, Theorem 8.1.8]). Two elements $a \otimes b$ and $c \otimes d$ in $A \otimes_S B$ are equal if and only if $(a, b) = (c, d)$ or there exist $a_1, a_2, \ldots, a_{n-1}$ in A, $b_1, b_2, \ldots, b_{n-1}$ in B, $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1}$ in S such that

$$\begin{aligned} a &= a_1 s_1, & s_1 b &= t_1 b_1 \\ a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 \\ a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} \qquad (i = 2, \ldots, n-2) \\ a_{n-1} t_{n-1} &= c s_n, & s_n b_{n-1} &= d. \end{aligned}$$

Theorem 5.2.2. Let U be a submonoid of a commutative monoid S, Then d is in $Dom(U, S)$ if and only if either $d \in U$ or there exists a zigzag in S over U with value d.

Proof. To prove the theorem, we, by Result 1.4.5, essentially show that if $d \in S$, then $d \in Dom(U, S)$ if and only if $d \otimes 1 = 1 \otimes d$ in $A = S \otimes_U S$, where 1 is the identity of S. So let us suppose first that $d \in S$ and $d \otimes 1 = 1 \otimes d$ in $A = S \otimes_U S$. Then, by Result 5.2.1, we have

$$\begin{aligned} d &= a_1 s_1, & s_1 &= t_1 b_1 \\ a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 \end{aligned}$$

$$a_i t_i = a_{i+1} s_{i+1}, \qquad s_{i+1} b_i = t_{i+1} b_{i+1} \qquad (i = 2, \ldots, n-2)$$

$$a_{n-1} t_{n-1} = s_n, \qquad s_n b_{n-1} = d;$$

where $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in S$ and $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1} \in U$

Let T be a semigroup and let $\alpha, \beta : S \to T$ be homomorphisms agreeing on U; i.e.

$$\alpha \mid U = \beta \mid U$$

Now, by using zigzag equations, we have

$$\begin{aligned}
\alpha(d) &= \alpha(a_1 s_1) \\
&= \alpha(a_1)\alpha(s_1) \\
&= \alpha(a_1)\beta(s_1) \\
&= \alpha(a_1)\beta(t_1 b_1) \\
&= \alpha(a_1)\beta(t_1)\beta(b_1) \\
&= \alpha(a_1)\alpha(t_1)\beta(b_1) \\
&= \alpha(a_1 t_1)\beta(b_1) \\
&= \alpha(a_2 s_2)\beta(b_1) \\
&\vdots \\
&= \alpha(a_i t_i)\beta(b_i) \\
&= \alpha(a_{i+1} s_{i+1})\beta(b_i) \\
&= \alpha(a_{i+1})\alpha(s_{i+1})\beta(b_i) \\
&= \alpha(a_{i+1})\beta(s_{i+1})\beta(b_i) \\
&= \alpha(a_{i+1})\beta(s_{i+1} b_i) \\
&\vdots \\
&= \alpha(a_{n-1} t_{n-1})\beta(b_{n-1}) \\
&= \alpha(s_n)\beta(b_{n-1}) \\
&= \beta(s_n)\beta(b_{n-1}) \\
&= \beta(s_n b_{n-1}) \\
&= \beta(d) \\
\Rightarrow \alpha(d) &= \beta(d).
\end{aligned}$$

Therefore, $d \in Dom(U, S)$.

To prove the converse, we first show that for a commutative monoid, the equivalence relation τ is a congruence; i.e.

$$(a, b)\tau(c, d)\tau = (ac, bd)\tau.$$

For this, we have to show that τ is compatible; i.e.

if $(a, b)\tau = (c, d)\tau$ and $(a', b')\tau = (c', d')\tau$, then $((a, b)(a', b'))\tau = ((c, d)(c', d'))\tau$.

Since $a \otimes b = c \otimes d$, by Result 5.2.1, we have

$$\begin{aligned} a &= a_1 s_1, & s_1 b &= t_1 b_1 & & \\ a_1 t_1 &= a_2 s_2, & s_2 b_1 &= t_2 b_2 & & \\ a_i t_i &= a_{i+1} s_{i+1}, & s_{i+1} b_i &= t_{i+1} b_{i+1} & & (i = 2, \ldots, n-2) \\ a_{n-1} t_{n-1} &= c s_n, & s_n b_{n-1} &= d; & & (A) \end{aligned}$$

for some $a_1, a_2, \ldots, a_{n-1}, b_1, b_2, \ldots, b_{n-1} \in S$ and $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_{n-1} \in U$.

Similarly, as $a' \otimes b' = c' \otimes d'$, we have

$$\begin{aligned} a' &= a'_1 s'_1, & s'_1 b' &= t'_1 b'_1 & & \\ a'_1 t'_1 &= a'_2 s'_2, & s'_2 b'_1 &= t'_2 b'_2 & & \\ a'_i t'_i &= a'_{i+1} s'_{i+1}, & s'_{i+1} b'_i &= t'_{i+1} b'_{i+1} & & (i = 2, \ldots, n-2) \\ a'_{n-1} t'_{n-1} &= c' s'_n, & s'_n b'_{n-1} &= d'; & & (B) \end{aligned}$$

for some $a'_1, a'_2, \ldots, a'_{n-1}, b'_1, b'_2, \ldots, b'_{n-1} \in S$ and $s'_1, s'_2, \ldots, s'_n, t'_1, t'_2, \ldots, t'_{n-1} \in U$.

Now, from equations (A) and (B), we have

$$\begin{aligned} aa' &= (a_1 s_1)(a'_1 s'_1), & (s_1 b)(s'_1 b') &= (t_1 b_1)(t'_1 b'_1) \\ (a_1 t_1)(a'_1 t'_1) &= (a_2 s_2)(a'_2 s'_2), & (s_2 b_1)(s'_2 b'_1) &= (t_2 b_2)(t'_2 b'_2) \end{aligned}$$

$$(a_it_i)(a'_it'_i) = (a_{i+1}s_{i+1})(a'_{i+1}s'_{i+1}), \qquad (s_{i+1}b_i)(s'_{i+1}b'_i) = (t_{i+1}b_{i+1})(t'_{i+1}b'_{i+1})$$
$$(i = 2, \ldots, n-2)$$
$$(a_{n-1}t_{n-1})(a'_{n-1}t'_{n-1}) = (cs_n)(c's'_n), \qquad (s_nb_{n-1})(s'_nb'_{n-1}) = dd'.$$

Since, in the above system of equalities all members belong to S, so, by using commutativity of S, we have

$$aa' = (a_1a'_1)(s_1s'_1), \qquad (s_1s'_1)(bb') = (t_1t'_1)(b_1b'_1)$$
$$(a_1a'_1)(t_1t'_1) = (a_2a'_2)(s_2s'_2), \qquad (s_2s'_2)(b_1b'_1) = (t_2t'_2)(b_2b'_2)$$
$$(a_ia'_i)(t_it'_i) = (a_{i+1}a'_{i+1})(s_{i+1}s'_{i+1}), \qquad (s_{i+1}s'_{i+1})(b_ib'_i) = (t_{i+1}t'_{i+1})(b_{i+1}b'_{i+1})$$
$$(i = 2, \ldots, n-2)$$
$$(a_{n-1}a'_{n-1})(t_{n-1}t'_{n-1}) = (cc')(s_ns'_n), \qquad (s_ns'_n)(b_{n-1}b'_{n-1}) = dd';$$

where $a_1a'_1, a_2a'_2, \ldots, a_{n-1}a'_{n-1}, b_1b'_1, b_2b'_2, \ldots, b_{n-1}b'_{n-1} \in S$ and $s_1s'_1, s_2s'_2, \ldots, s_ns'_n$, $t_1t'_1, t_2t'_2, \ldots, t_{n-1}t'_{n-1} \in U$.

Thus, by Result 5.2.1, we have

$aa' \otimes bb' = cc' \otimes dd' \Rightarrow (aa', bb')\tau = (cc', dd')\tau \Rightarrow ((a,b)(a',b'))\tau = ((c,d)(c',d'))\tau \Rightarrow \tau$ is a congruence.

Now define $\alpha : S \to S \times A$ and $\beta : S \to S \times A$
by

$$\alpha(s) = (s, s \otimes 1), \ \beta(s) = (s, 1 \otimes s).$$

Then α, β are, clearly, semigroup morphisms.

Since, $u \otimes 1 = 1 \otimes u$, we have

$\alpha(u) = \beta(u)$, for all $u \in U$.

Therefore $\alpha(d) = \beta(d)$

$\Rightarrow (d, d \otimes 1) = (d, 1 \otimes d)$

$\Rightarrow d \otimes 1 = 1 \otimes d.$

This completes the proof of the theorem. □

Thus we have the following:

Theorem 5.2.3. If U is a submonoid of a commutative monoid S, then d is in $Dom(U, S)$ if and only if either $d \in U$ or there exists a zigzag in S over U with value d. □

It may easily be verified that the arguments employed by Howie [51] in proving Theorems 8.3.4 to 8.3.5 work through to complete the proof for the following Isbell's Zigzag Theorem for the category of all commutative semigroups.

Theorem 5.2.4. Let U be a subsemigroup of a commutative semigroup S. Then $d \in Dom(U, S)$ if and only if either $d \in U$ or there exists a zigzag in S over U with value d. □

§ 5.3. ABSOLUTELY CLOSED SEMIGROUPS

Most notable results in this area have been to the effect that certain classes of semigroups consist entirely of absolutely closed semigroups or entirely of saturated semigroups. For example, it was shown by Howie and Isbell [54] that right simple semigroups, finite monogenic semigroups and inverse semigroups are absolutely closed. Scheiblich and Moore [81] showed that the total transformation semigroup $\mathcal{T}_X$, on any set X, is absolutely closed; this result is also proved by Hall [28], whose proof also works for the semigroup of partial transformations on a set.

In [30], Higgins proved that the class of all generalized inverse semigroups is saturated. Since the class of all regular medial semigroups is contained in the class of all generalized semigroups, the class of all regular medial semigroups is saturated. We, now, extend this result and show that a subclass of the class of all regular medial semigroups is absolutely closed.

Definition 5.3.1 ([9]). A semigroup S is said to be *weakly separative* if for $a, b \in S$, $asa = asb = bsa = bsb$ for all $s \in S$ implies $a = b$.

Theorem 5.3.2. Let $\mathcal{C}$ be the class of all regular medial semigroups satisfying the identity $x^{n+1} = x^n y$ for some positive integer $n \geq 2$. Then $\mathcal{C}$ is absolutely closed.

Result 5.3.3 ([84, Lemma 3]). Every regular semigroup is weakly separative.

First we prove the following lemma.

Lemma 5.3.4. Let S be a weakly separative medial semigroup satisfying the identity $x^{n+1} = x^n y$ for some positive integer $n \geq 2$. Then $xy = x^2$ for all $x, y \in S$.

Proof. For any $x, y \in S$ and $\forall s \in S$, we have

$$\begin{aligned}
(x^n)s(x^n) &= (x^n)s(x^n) \\
&= x^{n-1}(xs)x^n \\
&= x^{n-1}(sx)x^n && \text{(as } S \text{ is medial)} \\
&= x^{n-1}sx^{n+1} \\
&= x^{n-1}sx^n y && \text{(by property of } S\text{)} \\
&= x^{n-1}(sx)x^{n-1}y \\
&= x^{n-1}(xs)x^{n-1}y && \text{(as } S \text{ is medial)}
\end{aligned}$$

$$= \quad (x^n)s(x^{n-1}y).$$

Similarly, we may prove that

$$(x^n)s(x^n) = (x^{n-1}y)s(x^n) = (x^{n-1}y)s(x^{n-1}y).$$

Since $(x^n)s(x^n) = (x^n)s(x^{n-1}y) = (x^{n-1}y)s(x^n) = (x^{n-1}y)s(x^{n-1}y)$ and, as S is weakly separative, we have $x^n = x^{n-1}y$.

Repeating the above argument $n-2$ times, we will get $xy = x^2$

Dually, we may prove the following:

Lemma 5.3.5. Let S be a weakly separative medial semigroup satisfying the identity $x^{n+1} = yx^n$ for some positive integer $n \geq 2$. Then $yx = x^2$ for all $x, y \in S$.

Using Result 5.3.3 and Lemmas 5.3.4 and 5.3.5, we have the following:

Remark 5.3.6. Let S be a regular medial semigroup satisfying the identity $x^{n+1} = x^n y$ $[x^{n+1} = yx^n]$ for some positive integer $n \geq 2$. Then $xy = x^2$ $[yx = x^2]$ for all $x, y \in S$.

Proof of the main theorem. Let U be a regular medial subsemigroup such that $x^{n+1} = x^n y$ and S be any semigroup contains U. Take any $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.
Now

$$\begin{aligned} d &= a_0 y_1 \\ &= x_1 a_1 y_1 && \text{(by zigzag equations)} \\ &= x_1 a_1 a_1' a_1 y_1 && (\text{as } a_1 \in U) \end{aligned}$$

$$
\begin{aligned}
&= x_1a_1a_1'a_2y_2 && \text{(by zigzag equations as } a_1y_1 = a_2y_2\text{)}\\
&= x_1a_1a_1'a_2a_2'a_2y_2 && \text{(since } a_2 \in U\text{)}\\
&= x_1a_1a_1'a_2e_1y_2 && \text{(where } e_1 = a_2'a_2\text{)}\\
&= x_1a_1a_1'a_2{e_1}^2y_2 && \text{(since } e_1 \in E(S)\text{)}\\
&= x_1a_1a_1'a_2e_1a_3y_2 && \text{(as } x^2 = xy\text{)}\\
&= x_1a_1a_1'a_2(a_3y_2) && \text{(as } e_1 = a_2'a_2\text{)}\\
&\vdots\\
&= x_1a_1a_1'a_2a_4\cdots a_{2m-6}a_{2m-4}(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'a_2a_4\cdots a_{2m-6}a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_1a_1a_1'a_2a_4\cdots a_{2m-6}a_{2m-4}(a_{2m-2}a_{2m-2}'a_{2m-2})y_m && \text{(since } a_{2m-2} \text{ is regular)}\\
&= x_1a_1a_1'a_2a_4\cdots a_{2m-6}a_{2m-4}a_{2m-2}(a_{2m-2}'a_{2m-2})^2y_m && \text{(since } a_{2m-2}'a_{2m-2} \in E(S)\text{)}\\
&= x_1a_1a_1'a_2a_4\cdots a_{2m-6}a_{2m-4}a_{2m-2}(a_{2m-2}'a_{2m-2})a_{2m-1}y_m && \text{(since } x^2 = xy\text{)}\\
&= a_0a_1'a_2a_4\cdots a_{2m-6}a_{2m-4}a_{2m-2}(a_{2m-2}'a_{2m-2})a_{2m} \in U && \text{(by zigzag equations)}\\
&\Rightarrow\ d \in U.
\end{aligned}
$$

Hence $Dom(U, S) = U$. □

Dually, we may prove the following:

Theorem 5.3.7. Let $\mathcal{C}$ be the class of all regular medial semigroup satisfying the identity $x^{n+1} = yx^n$ for some positive integer $n \geq 2$. Then $\mathcal{C}$ is absolutely closed. □

Theorem 5.3.8. All globally idempotent commutative semigroups satisfying the identity $ax = axa[xa = axa]$ are absolutely closed.

Let us now assume that U be a globally idempotent commutative semigroup satisfying the identity $ax = axa[xa = axa]$ and let U be a subsemigroup in every properly containing semigroup S.

Lemma 5.3.9. For all $a \in U$ and $x, y \in S \setminus U$, we have $xay = xa^2y$.

Proof. Since $a \in U$, we have

$$a = c_1c_2\cdots c_n \qquad \forall\ c_1, c_2, \cdots, c_n \in U,$$

Now

$$\begin{aligned}
xay &= xc_1c_2\cdots c_ny \\
&= xc_1(c_2\cdots c_n)y \\
&= xc_1(c_2\cdots c_n)c_1y \qquad (\text{since } c_1, (c_2\cdots c_n) \in U) \\
&= xc_1(c_2\cdots c_n)c_1(c_2\cdots c_n)y \qquad (\text{since } (c_2\cdots c_n), c_1 \in U) \\
&= x(c_1c_2\cdots c_n)(c_1c_2\cdots c_n)y \\
&= xa^2y
\end{aligned}$$

This proves Lemma 5.3.9.

Now to complete the proof of the theorem, we take any $d \in S \setminus U$, and let (1) be a zigzag for d over U. Then

$$d = x_1a_1y_1$$

$$
\begin{aligned}
&= x_1 a_1^2 y_1 \\
&= x_1 a_1 a_2 y_2 \\
&= x_1 a_2 a_1 y_2 \\
&= x_2 a_3 a_1 y_2 \\
&= x_2 a_3^2 a_1 y_2 \\
&= x_1 a_2 a_1 a_3 y_2 \\
&= x_2 a_3 a_1 a_3 y_2 \\
&\vdots \\
&= x_m a_{2m-1} a_1 a_3 \cdots a_{2m-3} a_{2m-1} y_m \\
&= x_{m-1} a_{2m-2} a_1 a_3 \cdots a_{2m-3} a_{2m} \\
&= x_{m-1} a_{2m-3} a_{2m-2} a_1 a_3 \cdots a_{2m-5} a_{2m} \\
&= x_1 a_1 a_2 a_4 \cdots a_{2m-2} a_{2m} \\
&= a_0 a_2 \cdots a_{2m} \\
&= \prod_{i=0}^{m} a_{2i} \in U.
\end{aligned}
$$

CHAPTER 6

EMBEDDING OF SPECIAL SEMIGROUP AMALGAMS

§ 6.1. INTRODUCTION

In this chapter, we study the classes of all those regular semigroups which have the special amalgamation property. In Section 6.2, we show that a regular subsemigroup of a semigroup satisfying some condition in the containing semigroup is closed in the containing semigroup. As immediate corollaries, we have got that the special semigroup amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$ within the class of left [right] quasinormal orthodox semigroups, $\mathcal{R}[\mathcal{L}]$-unipotent semigroups and left[right] Clifford semigroups is embeddable in a left [right] quasinormal orthodox semigroup, $\mathcal{R}[\mathcal{L}]$-unipotent semigroup and left[right] Clifford semigroup respectively. Further, we prove that every special amalgam of the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is strongly embeddable in the class of all left[right] semiregular orthodox semigroups which implies that every special amalgam of the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is strongly embeddable in the class of all quasi-inverse semigroups. Finally, we show that the classes of all WL[WR]-regular orthodox semigroups and that of all left[right] seminormal orthodox semigroups have special amalgamation property.

In Section 6.3, we have proved some results on closedness of the class of all bands in to the class of all semigroups satisfying some homotypical identities. Firstly, we prove that the class of all normal bands is closed within the class of all medial semigroups, generalizing the long known fact that the class of all normal bands is closed (see [80]). Finally, we prove that the class of all left[right] seminormal bands is closed within the class of all semigroups satisfying the identity $axy = axyay[yxa = yayxa]$. From this result, we got a corollary that the class of all left[right] seminormal bands is closed. This, generalizes the result on quasinormal bands.

§ 6.2. ON AMALGAMS OF REGULAR SEMIGROUPS

Howie and Isbell [54] had shown that each inverse semigroup is a special amalgamation base. Whether we can generalize or not this result in the class of all regular semigroups is an open problem. In this section, we have studied the special amalgamation property in some subclasses of regular semigroups which are wider or different classes than the class of inverse semigroups. Firstly, we show that special amalgam $\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}]$, satisfying some condition, where U is a regular subsemigroup of S, is strongly embeddable. Further, we prove that the class of all left[right] Clifford semigroups, $\mathcal{R}[\mathcal{L}]$-unipotent semigroups, left [right] quasinormal orthodox semigroups, WL[WR]-regular orthodox semigroups and the class of all left[right] seminormal orthodox semigroups have special amalgamation property. We do not know whether or not the class of all quasi-inverse semigroups has special amalgamation property. However, we are able to show that the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is strongly embeddable in the class of all left[right] semiregular orthodox semigroups, which implying that the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is strongly embeddable in the class of all quasi-inverse semigroups.

Definition 6.2.1. A semigroup S is said to be *left[right] Clifford* if S is regular and $aS \subseteq Sa[Sa \subseteq aS]$ for all $a \in S$.

Characterization of a left Clifford semigroups may be stated as follows.

Result 6.2.2. Let S be a semigroup and let $E(S)$, the set of all idempotents of S, be a band. Then the following statements on S are equivalent:

(i) S is a left Clifford semigroup;

(ii) $(\forall e \in E(S))\ eS \subseteq Se$;

(iii) $(\forall e \in E(S))\ (\forall a \in S) eae = ea$.

Characterization of a right Clifford semigroup may be stated dually.

Definition 6.2.3. A *$\mathcal{R}$-unipotent* semigroup S is a regular semigroup whose set of idempotents form a left regular band (i.e. $E(S)$ is a subsemigroup satisfying the identity $efe = ef$).

$\mathcal{L}$-unipotent semigroups are defined dually. Structure theorems for $\mathcal{R}$-unipotent semigroups may be found in [90] and [91].

Definition 6.2.4. A *left[right] quasi normal orthodox* semigroup S is a regular semigroup whose idempotents form a left[right] quasinormal band (i.e. $efg = efeg[gfe = gefe]$ $\forall e, f, g \in E(S)$, see [72] for more details).

The class of all left quasinormal orthodox semigroups contains both the classes of $\mathcal{R}$-unipotent semigroups and that of generalized inverse semigroups. Dually the class of all right quasinormal orthodox semigroups contains the class of $\mathcal{L}$-unipotent semigroups and that of generalized inverse semigroups.

The following result is a part of left-right dual of Theorem 1 in [90].

Result 6.2.5. Let S be a regular semigroup. Then the following statements are equivalent.

(a) S is $\mathcal{R}$-unipotent;

(b) $(\forall e \in E)(\forall a \in S)(\forall a' \in V(a))$ $ae = aea'a$.

We require the following well known properties of a left quasinormal orthodox semigroup. In the following, we shall denote by a', u', etc. as arbitrary inverses of a, u, etc.

Result 6.2.6. Let S be a left quasinormal orthodox semigroup. Let $a \in S$ and e be an idempotent of S.

(i) If a' is an inverse of a, then aea' and $a'ea$ are idempotents.

(ii) $(\forall e, f \in E)(\forall a \in S)(\forall a' \in V(a))$ $aef = aea'af$.

The above result for a right quasinormal orthodox semigroup may be stated dually.

Definition 6.2.7. A *quasi-inverse semigroup* S is a regular semigroup whose set of idempotents forms a regular band i.e. $efge = efege$ $\forall e, f, g \in E(S)$.

Definition 6.2.8. A *left[right] seminormal orthodox semigroup* S is a regular semigroup whose set of idempotents forms a left[right] seminormal band i.e. $efg = efgeg[gfe = gegfe]$ $\forall e, f, g \in E(S)$.

Definition 6.2.9. A *WL[WR] regular orthodox semigroup* S is a regular semigroup whose set of idempotents forms a *WL[WR]- regular band* i.e. $efg = efege[fge = efege]$ $\forall e, f, g \in E(S)$.

It is clear that a class of all R[L]-unipotent semigroups is contained in the class of all WL[WR]- regular orthodox semigroups which in turn is contained in the class of all quasi-inverse semigroups. The class of all seminormal orthodox semigroups and that of all quasi-inverse semigroups are two different classes within the class of all regular semigroups. These classes of semigroups contain both the class of all left[right] quasi-normal orthodox semigroups as well as the class of all R[L]-unipotent semigroups.

We require the following well known properties of a quasi-inverse semigroup, WL[WR] regular orthodox semigroup and left[right] seminormal orthodox semigroup.

Result 6.2.10. Let S be a semigroup with a band $E(S)$ of idempotents. Then the following statements on S are equivalent:

(i) S is a quasi-inverse semigroup;

(ii) $(\forall e, f \in E(S))(\forall a \in S)(\forall a' \in V(a))$ $aefa'a = aea'afa'a$.

Result 6.2.11. Let S be a semigroup with a band $E(S)$ of idempotents. Then the following statements on S are equivalent:

(i) S is a WL- regular orthodox semigroup;

(ii) $(\forall e, f \in E(S))(\forall a \in S)(\forall a' \in V(a))$ $aef = aea'afa'a$.

Characterization of WR- regular orthodox semigroup may be stated dually.

Result 6.2.12. Let S be a regular semigroup and $E(S)$ be a set of idempotents of S. Then the following statements are equivalent.

(i) S is a left seminormal orthodox semigroup;

(ii) $(\forall e, f \in E(S))(\forall a \in S)(\forall a' \in V(a))$ $aef = aefa'af$.

Dually, we may state the above result about a right seminormal orthodox semigroup.

Result 6.2.13. Let S be a regular semigroup. Let $a \in S$ and e be an idempotent of S. If a' is an inverse of a, then aea' and $a'ea$ are idempotents.

In [80], Scheiblich had proved, by using zigzag manipulations, that the class of all normal bands is closed. In [5], Alam and Khan generalized this result to the class of all left[right] regular bands. In [30], Higgins had shown that a generalized inverse subsemigroup of a semigroup satisfying a certain condition is closed in the containing semigroup.

Infact he proved the following Result:

Result 6.2.14 ([30, Proposition 3]). Let S be semigroup and U be a generalized inverse subsemigroup of S. For all $s, t \in S$ and all $e, f \in E(U)$ suppose that $seft = sfet$. Then U is closed in S.

Proof. Suppose $d \in Dom(U, S) \setminus U$. Then there exist a zigzag in S over U with value d. Suppose this zigzag has length one and so has the form $d = u_0 y = x u_1 y = x u_2$. Then

$$d = x u_1 y = (x u_1) u_1' (u_1 y) = u_0 u_1' u_2 \in U.$$

Therefore to complete the proof it is sufficient to show that any zigzag in S over U with value d and length $m > 1$ can be replaced by zigzag of length $m - 1$ with the same value. Hence suppose d has zigzag as in Result 1.6.1. Now we have

$$d = x_1 u_1 y_1 = (x_1 u_1) u_1' (u_1 y_1) = u_0 u_1' u_2 y_2.$$

Also

$$(x_2 u_3) u_2' u_1 u_1' u_2 = x_1 (u_2 u_2')(u_1 u_1') u_2 = x_1 (u_1 u_1')(u_2 u_2' u_2).$$

Therefore

$$x_2 u_3 u_2' u_1 u_1' u_2 = (x_1 u_1) u_1' u_2 = u_0 u_1' u_2.$$

This calculation justifies the first two lines of the following zigzag of length $m - 1$:

$$\begin{aligned} d &= (u_0 u_1' u_2) y_2 \\ &= x_2 (u_3 u_2' u_1 u_1' u_2) y_2 \end{aligned}$$

$$
\begin{aligned}
&= x_2(u_3u_2'u_1u_1'u_2u_3'u_4)y_3 \\
&\vdots \\
&= x_{m-1}(u_{2m-3}u_{2m-4}'\cdots u_1u_1'\cdots u_{2m-3}'u_{2m-2})y_m \\
&= x_m(u_{2m-1}u_{2m-2}'\cdots u_1u_1'\cdots u_{2m-3}'u_{2m-2})y_m \\
&= x_m(u_{2m-1}u_{2m-2}'\cdots u_1u_1'\cdots u_{2m-1}'u_{2m})
\end{aligned}
$$

where the bracketed terms form the new zigzag's spine.

The argument at the i^{th} stage is as follows. Given

$$d = x_i(u_{2i-1}u_{2i-2}'\cdots u_1u_1'\cdots u_{2i-2})y_i$$

then we also have

$$u_{2i-1}u_{2i-2}'\cdots u_1u_1'\cdots u_{2i-2}y_i$$

$$= u_{2i-1}[u_{2i-1}'u_{2i-1}][u_{2i-2}'\cdots u_1u_1'\cdots u_{2i-2}]y_i.$$

The bracketed terms are idempotents: the latter from Result 1.2.20 and induction. Then since idempotents commute with in products this equals

$$u_{2i-1}u_{2i-2}'\cdots u_1u_1'\cdots u_{2i-2}u_{2i-1}'u_{2i-1}y_i$$

$$= u_{2i-1}u_{2i-2}'\cdots u_1u_1'\cdots u_{2i-1}'u_{2i}y_{i+1}.$$

Therefore $d = x_i(u_{2i-1}u_{2i-2}'\cdots u_1u_1'\cdots u_{2i})y_{i+1}$, which is the next line of the zigzag. An argument for two consecutive lines involving the same y_i is dual to the above. This completes the proof. □

We extend this result to a regular subsemigroup of a semigroup satisfying some conditions in the containing semigroup. Next, we prove a similar result about a regular subsemigroup satisfying some conditions in the containing regular semigroup. As immediate corollaries of this result, we have got that the classes of all left[right] quasinormal orthodox semigroups and that of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent], left[right] Clifford semigroups are closed within the classes of all left[right] quasinormal orthodox semigroups and that of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent], left[right] Clifford semigroups respectively.

The following theorem extends Result 6.2.14 to regular semigroups.

Theorem 6.2.15. Let S be a semigroup and U be a regular subsemigroup of S. If $se = ses$ $\forall s \in S$ and $\forall e \in E(U)$, then U is closed in S.

Proof. Take any $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag of minimal length m in S over U with value d. Now

$$
\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_1' a_1 y_1 && \text{(as } U \text{ is a regular semigroup)} \\
&= x_1 a_1 a_1' a_2 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_1' x_1 a_2 y_2 && \text{(as } x_1 \in S \text{ and } a_1 a_1' \in E(U)) \\
&= x_1 a_1 a_1' x_2 a_3 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_1' x_2 a_3 a_3' a_3 y_2 && \text{(as } U \text{ is a regular semigroup)} \\
&= x_1 a_1 a_1' x_1 a_2 a_3' a_3 y_2 && \text{(by the zigzag equations)} \\
&= x_1 a_1 a_1' a_2 a_3' a_3 y_2 && \text{(as } x_1 \in S \text{ and } a_1 a_1' \in E(U))
\end{aligned}
$$

$$
\begin{aligned}
&= a_0a_1'a_2a_3'(a_3y_2) && \text{(by the zigzag equations)}\\
&\vdots\\
&= a_0a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(as } x_1\in S \text{ and } a_1a_1'\in E(U))\\
&= x_1a_1a_1'x_2a_3a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_2a_3a_3'x_2a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(as } x_2\in S \text{ and } a_3a_3'\in E(U))\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'x_3a_6\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
& && \text{(as } x_3\in S \text{ and } a_5a_5'\in E(U))\\
&\vdots\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-2}a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-1}a_{2m-3}a_{2m-3}'(a_{2m-2}y_m) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_2a_3a_3'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}a_{2m-2}y_m\\
& && \text{(as } x_{m-1}\in S \text{ and } a_{2m-3}a_{2n-3}'\in E(U))\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_ma_{2m-1}y_m && \text{(by zigzag equations)}\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_ma_{2m-1}a_{2m-1}'a_{2m-1}y_m && \text{(as } U \text{ is regular)}\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}a_{2m-2}a_{2m-1}'a_{2m} && \text{(by zigzag equations)}
\end{aligned}
$$

$$
\begin{aligned}
&= x_1 a_1 a_1' \cdots x_{m-1} a_{2m-3} a_{2m-3}' a_{2m-2} a_{2m-1}' a_{2m} \\
&\qquad\qquad\qquad\qquad\qquad (\text{as } x_{m-1} \in S \text{ and } a_{2m-3} a_{2m-3}' \in E(U)) \\
&\vdots \\
&= x_1 a_1 a_1' x_1 a_2 a_3' a_4 a_5' \cdots a_{2m-4} a_{2m-3}' a_{2m-2} a_{2m-1}' a_{2m} \\
&= x_1 a_1 a_1' a_2 a_3' a_4 a_5' \cdots a_{2m-4} a_{2m-3}' a_{2m-2} a_{2m-1}' a_{2m} \qquad (\text{as } x_1 \in S \text{ and } a_1 a_1' \in E(U)) \\
&= a_0 a_1' a_2 a_3' a_4 a_5' \cdots a_{2m-4} a_{2m-3}' a_{2m-2} a_{2m-1}' a_{2m} \in U \qquad (\text{by zigzag equations})
\end{aligned}
$$

$\Rightarrow$ $d \in U$. Hence $Dom(U,S) = U$. □

Dually, we may prove the following:

Theorem 6.2.16. Let S be a semigroup and U be a regular subsemigroup of S. If $es = ses \;\; \forall s \in S$ and $\forall e \in E(U)$, then U is closed in S. □

The following theorem immediately shows that the class of all left[right] quasi-normal orthodox semigroups and the class of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent](left[right] Clifford) semigroups are closed within the class of all left[right] quasinormal orthodox semigroups and the class of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent](left[right] Clifford) semigroups respectively.

Theorem 6.2.17. Let S be any regular semigroup and U be any regular subsemigroup of S. If $efg = efeg \;\; \forall e \in E(S)$ and $\forall f, g \in E(U)$, then U is closed in S.

Proof. Suppose $d \in Dom(U,S) \setminus U$. Then, by Result 1.4.5, there exist a zigzag (1.4.1) in S over U with value d of minimal length m. Now

$$
\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 \qquad\qquad\qquad (\text{by zigzag equations})
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1a_1'a_1y_1 && \text{(as } U \text{ is a regular semigroup)}\\
&= x_1a_1a_1'a_2y_2 && \text{(by zigzag equations)}\\
&= x_1a_1a_1'a_2a_2'a_2y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= x_1x_1'x_1a_1a_1'a_2a_2'a_2y_2 && \text{(as } S \text{ is a regular semigroup)}\\
&= x_1a_1a_1'x_1'x_1a_2a_2'a_2y_2 && \text{(as } x_1'x_1 \in E(S) \text{ \& } a_1a_1', a_2a_2' \in E(U))\\
&= x_1a_1a_1'x_1'x_1a_2y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= x_1a_1a_1'x_1'x_2a_3y_2 && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_2a_3a_3'a_3y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= x_1a_1a_1'x_1'x_1a_2a_3'a_3y_2 && \text{(by the zigzag equations)}\\
&= x_1a_1a_1'x_1'x_1a_2a_2'a_2a_3'a_3y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= x_1a_1a_1'a_2a_2'a_2a_3'a_3y_2 && \text{(as } x_1'x_1 \in E(S) \text{ \& } a_1a_1', a_2a_2' \in E(U))\\
&= x_1a_1a_1'a_2a_3'a_3y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= a_0a_1'a_2a_3'(a_3y_2) && \text{(by the zigzag equations)}\\
&\vdots\\
&= a_0a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1a_1'x_1'x_1a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(as } x_1'x_1\in E(S)\ \&\ a_1a_1',a_2a_2'\in E(U))\\
&= x_1a_1a_1'x_1'x_2a_3a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) \qquad \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_2a_3a_3'x_2'x_2a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(as } x_2'x_2\in E(S)\ \&\ a_3a_3',a_4a_4'\in E(U))\\
&= x_1a_1a_1'x_1'x_2a_3a_3'x_2'x_3a_5a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) \qquad \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_2a_3a_3'x_2'x_3a_5a_5'x_3'x_3a_6\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(as } x_3'x_3\in E(S)\ \&\ a_5a_5',a_6a_6'\in E(U))\\
&\ \vdots\\
&= x_1a_1a_1'x_1'x_2a_3a_3'\cdots x_{m-2}a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'x_1'x_2a_3a_3'\cdots x_{m-1}a_{2m-3}a_{2m-3}'(a_{2m-2}y_m) \qquad \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_2a_3a_3'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}y_m\\
&\qquad\qquad\qquad\qquad\qquad\text{(as } x_{m-1}'x_{m-1}\in E(S)\ \&\ a_{2m-3}a_{2m-3}',a_{2m-2}a_{2m-2}'\in E(U))\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_ma_{2m-1}y_m \quad \text{(by zigzag equations)}\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_ma_{2m-1}a_{2m-1}'a_{2m-1}y_m \qquad \text{(as } U \text{ is regular)}\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-1}'a_{2m} \qquad \text{(by zigzag equations)}\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}\\
&\qquad\qquad\qquad\qquad\qquad\text{(as } x_{m-1}'x_{m-1}\in E(S)\ \&\ a_{2m-3}a_{2m-3}',a_{2m-2}a_{2m-2}'\in E(U))\\
&\ \vdots\\
&= x_1a_1a_1'x_1'x_1a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}
\end{aligned}
$$

$$= x_1a_1a_1'a_2a_3'a_4a_5' \cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}$$

(as $x_1'x_1 \in E(S)$ & $a_1a_1', a_2a_2' \in E(U)$)

$$= a_0a_1'a_2a_3'a_4a_5' \cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} \in U \qquad \text{(by zigzag equations)}$$

$\Rightarrow d \in U$. Hence $Dom(U,S) = U$. □

Following corollaries are easy consequences of Theorem 6.2.17, Result 6.2.6, Result 6.2.5 and Result 6.2.2.

Corollary 6.2.18. If U is a left[right] quasinormal orthodox subsemigroup of a left[right] quasinormal orthodox semigroup S, then U is closed in S.

Corollary 6.2.19. If U is a $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent] subsemigroup of a $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent] semigroup S, then U is closed in S.

Corollary 6.2.20. If U is a left[right] Clifford subsemigroup of a left[right] Clifford semigroup S, then U is closed in S.

Above corollaries may also be stated as follows:

Corollary 6.2.21. Let U be a left [right] quasinormal orthodox ($\mathcal{R}[\mathcal{L}]$-unipotent, left[right] Clifford) subsemigroup of a left [right] quasinormal orthodox ($\mathcal{R}[\mathcal{L}]$-unipot -ent, left[right] Clifford) semigroup S. Let S' be a left [right] quasinormal orthodox ($\mathcal{R}[\mathcal{L}]$-unipotent, left[right] clifford) semigroup disjoint from S and let $\alpha : S \to S'$ be an isomorphism. Let $P = S *_U S'$, the free product of the amalgam

$$\mathcal{U} = [\{S, S'\};\ U;\ \{i, \alpha \mid U\}],$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P, then

$$S\mu \cap S'\mu' = U\mu.$$

Therefore the amalgam $\mathcal{U}$ is embeddable in a left[right] quasinormal orthodox ($\mathcal{R}[\mathcal{L}]$-unipotent, left[right] Clifford) semigroup.

In [2], we have shown that the classes of all quasinormal orthodox semigroups and $\mathcal{R}[\mathcal{L}]$-unipotent semigroups are closed. Therefore, it is natural to ask whether the classes of all left[right] quasinormal orthodox semigroups and that of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups are closed within the classes of all quasi-inverse semigroups and that of all left[right] semiregular orthodox semigroups respectively. In the forthcoming theorem, we show that the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is closed within the class of all left[right] semiregular orthodox semigroups. As a corollary, we got that the class of all $\mathcal{R}[\mathcal{L}]$-unipotent semigroups is closed within the class of all quasi-inverse semigroups. However, the closedness of the class of all quasinormal orthodox semigroups in the class of all quasi-inverse semigroups and left[right] semiregular orthodox semigroups is not yet known. Further, we have shown that the class of all WL[WR]-regular orthodox semigroups and the class of all left[right] seminormal orthodox semigroups have special amalgamation property.

Lemma 6.2.22. Let U be a $\mathcal{R}$-unipotent semigroup and S be any left semiregular orthodox semigroup such that U be a subsemigroup of S. If for $d \in Dom(U,S) \setminus U$ and (1.4.1) be a zigzag in S over U of minimal length m, then

$$(\prod_{i=1}^{j-1}(a_{2i-2}a'_{2i-1}))(a_{2j-3}y_{j-1}) = (\prod_{i=1}^{j}(a_{2i-2}a'_{2i-1}))(a_{2j-1}y_j) \quad (\forall\ j = 2, \cdots, m)$$

Proof. Now

$$\begin{aligned}
&(\prod_{i=1}^{j-1}(a_{2i-2}a'_{2i-1}))(a_{2j-3}y_{j-1}) \\
&= a_0a'_1a_2a'_3(\prod_{i=3}^{j-1}(a_{2i-2}a'_{2i-1}))(a_{2j-3}y_{j-1}) \\
&= x_1a_1a'_1a_2a'_3(\prod_{i=3}^{j-1}(a_{2i-2}a'_{2i-1}))(a_{2j-2}y_j) && \text{(by zigzag equations)} \\
&= x_1(x'_1x_1a_1a'_1a_2a'_2)a_2a'_3(\prod_{i=3}^{j-1}(a_{2i-2}a'_{2i-1}))a_{2j-2}y_j && \text{(since } x_1 \text{ and } a_2 \text{ are regular)}
\end{aligned}$$

$$= \quad x_1(x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2a_2'a_1a_1'a_2a_2')a_2a_3'(\prod_{i=3}^{j-1}(a_{2i-2}a_{2i-1}'))a_{2j-2}y_j$$

(since $x_1'x_1, a_1a_1', a_2a_2' \in E(S)$)

$$= \quad x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_2a_3a_2'a_1a_1'a_2a_2'a_2a_3'a_4a_5'(\prod_{i=4}^{j-1}(a_{2i-2}a_{2i-1}'))a_{2j-2}y_j$$

(by zigzag equations)

$$= \quad x_1a_1a_1'a_2a_2'x_1'x_2a_3a_2'a_1a_1'a_2a_2'a_2a_3'a_4a_5'(\prod_{i=4}^{j-1}(a_{2i-2}a_{2i-1}'))a_{2j-2}y_j$$

(since x_1 is regular)

$$= \quad z_1x_2(x_2'x_2a_3a_2'a_1a_1'a_2a_3'a_4a_4')a_4a_5'(\prod_{i=4}^{j-1}(a_{2i-2}a_{2i-1}'))a_{2j-2}y_j$$

(where $z_1 = x_1a_1a_1'a_2a_2'x_1'$ and as x_2, a_2, a_4 are regular)

$$= \quad z_1x_2(x_2'x_2e_1a_4a_4')a_4a_5'(\prod_{i=4}^{j-1}(a_{2i-2}a_{2i-1}'))a_{2j-2}y_j$$

(where $e_1 = a_3a_2'a_1a_1'a_2a_3' \in E(U)$)

$$= \quad z_1x_2(x_2'x_2e_1a_4a_4'x_2'x_2a_4a_4'fa_4a_4')a_4a_5'(\prod_{i=4}^{j-1}(a_{2i-2}a_{2i-1}'))a_{2j-2}y_j$$

(since $x_2'x_2, f, a_4a_4' \in E(S)$)

$\vdots$

$$= \quad z_1x_2\cdots x_{j-2}(x_{j-2}'x_{j-2}e_2a_{2j-4}a_{2j-4}')a_{2j-4}a_{2j-3}'a_{2j-2}y_j$$

(where $e_2 = a_{2j-5}a_{2j-6}'a_{2j-7}\cdots a_3a_2'a_1a_1'a_2a_3'\cdots a_{2j-7}'a_{2j-6}a_{2j-5}' \in E(U)$)

$$= \quad z_1x_2x_2'x_2\cdots x_{j-2}(x_{j-2}'x_{j-2}e_2a_{2j-4}a_{2j-4}')a_{2j-4}z_2 \qquad \text{(where } z_2 = a_{2j-3}'a_{2j-2}y_j\text{)}$$

$$= \quad z_1x_2\cdots x_{j-2}(x_{j-2}'x_{j-2}e_2a_{2j-4}a_{2j-4}'x_{j-2}'x_{j-2}a_{2j-4}a_{2j-4}'e_2a_{2j-4}a_{2j-4}')a_{2j-4}z_2$$

(since $x_{j-2}'x_{j-2}, e_2a_{2j-4}, a_{2j-4}' \in E(S)$)

$$= \quad z_1x_2\cdots x_{j-2}(x_{j-2}'x_{j-2}e_2a_{2j-4}a_{2j-4}'x_{j-2}'x_{j-1}a_{2j-3}a_{2j-4}'e_2a_{2j-4}a_{2j-4}')a_{2j-4}z_2$$

(as $x_{j-2}a_{2j-4} = x_{j-1}a_{2j-3}$ by zigzag equations)

$$= z_1x_2\cdots x_{j-2}(x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-1}a_{2j-3}a'_{2j-4}e_2a_{2j-4}a'_{2j-4})a_{2j-4}a'_{2j-3}a_{2j-2}y_j \quad \text{(where } z_2 = a'_{2j-3}a_{2j-2}y_j)$$

$$= z_1x_2\cdots x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-1}a_{2j-3}a'_{2j-4}e_2a_{2j-4}a'_{2j-3}a_{2j-2}y_j$$
(as x_{j-2} and a_{2j-4} are regular)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}a_{2j-3}a'_{2j-4}e_2a_{2j-4}a'_{2j-3})a_{2j-2}y_j$$
(where $z_3 = x_{j-2}x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-1}$)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2})a_{2j-2}y_j$$
(where $e_3 = a_{2j-3}a'_{2j-4}e_2a_{2j-4}a'_{2j-3} \in E(U)$ and a_{2j-2} is regular)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2}x'_{j-1}x_{j-1}a_{2j-2}a'_{2j-2}e_3a_{2j-2}a'_{2j-2})a_{2j-2}y_j$$
(since $x'_{j-1}x_{j-1}, e_3, a_{2j-2}a'_{2j-2} \in E(S)$)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2}x'_{j-1}x_ja_{2j-1}(a'_{2j-1}a_{2j-1}a'_{2j-2}e_3a_{2j-2})y_j$$
(as a_{2j-1}, a_{2j-2} are regular)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2}x'_{j-1}x_ja_{2j-1}(a'_{2j-1}a_{2j-1}e_4)y_j$$
(where $e_4 = a'_{2j-2}e_3a_{2j-2} \in E(U)$)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2}x'_{j-1}x_ja_{2j-1}(a'_{2j-1}a_{2j-1}e_4a'_{2j-1}a_{2j-1})y_j$$
(since $a'_{2j-1}a_{2j-1}, a'_{2j-2}e_3a_{2j-2} \in E(U)$)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2}x'_{j-1}x_ja_{2j-1}e_4a'_{2j-1}a_{2j-1}y_j \quad \text{(as } a_{2j-1} \text{ is regular)}$$

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2}x'_{j-1}x_{j-1}a_{2j-2}e_4a'_{2j-1}a_{2j-1}y_j$$
(as $x_ja_{2j-1} = x_{j-1}a_{2j-2}$ by zigzag equations)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2}x'_{j-1}x_{j-1}a_{2j-2}a'_{2j-2}e_3a_{2j-2}a'_{2j-1}a_{2j-1}y_j$$

(where $e_4 = a'_{2j-2}e_3a_{2j-2} \in E(U)$)

$$= z_1x_2\cdots z_3(x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-2})a_{2j-2}a'_{2j-1}a_{2j-1}y_j$$

(since $x'_{j-1}x_{j-1}, e_3, a_{2j-2}a'_{2j-2} \in E(S)$)

$$= z_1x_2\cdots x_{j-2}x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-1}x'_{j-1}x_{j-1}e_3a_{2j-2}a'_{2j-1}a_{2j-1}y_j$$

(where $z_3 = x_{j-2}x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-1}$ and as a_{2j-2} is regular)

$$= z_1x_2\cdots x_{j-2}x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-1}e_3a_{2j-2}a'_{2j-1}(a_{2j-1}y_j)$$

(since x_{j-1} is regular)

$$= z_1x_2\cdots x_{j-2}x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-1}a_{2j-3}a'_{2j-4}e_2a_{2j-4}a'_{2j-3}a_{2j-2}a'_{2j-1}$$
$$(a_{2j-1}y_j)$$ (where $e_3 = a_{2j-3}a'_{2j-4}e_2a_{2j-4}a'_{2j-3} \in E(U)$)

$$= z_1x_2\cdots x_{j-2}(x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4}x'_{j-2}x_{j-2}a_{2j-4}a'_{2j-4}e_2a_{2j-4}a'_{2j-4})a_{2j-4}a'_{2j-3}$$
$$a_{2j-2}a'_{2j-1}(a_{2j-1}y_j)$$

(as $x_{j-1}a_{2j-3} = x_{j-2}a_{2j-4}$ by zigzag equations and a_{2j-4} is regular)

$$= z_1x_2\cdots x_{j-2}(x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4})(\prod_{i=j-2}^{j-1}(a_{2i-2}a'_{2i-1}))(a_{2j-1}y_j)$$

(since $x'_{j-2}x_{j-2}, e_2, a_{2j-4}a'_{2j-4} \in E(S)$)

$$= x_1a_1a'_1a_2a'_2x'_1x_2\cdots x_{j-2}(x'_{j-2}x_{j-2}e_2a_{2j-4}a'_{2j-4})(\prod_{i=j-2}^{j-1}(a_{2i-2}a'_{2i-1}))(a_{2j-1}y_j)$$

(where $z_1 = x_1a_1a'_1a_2a'_2x'_1$)

$$\vdots$$

$$= x_1(x'_1x_1a_1a'_1a_2a'_2x'_1x_1a_2a'_2a_1a'_1a_2a'_2)a_2a'_3(\prod_{i=3}^{j-1}(a_{2i-2}a'_{2i-1}))(a_{2j-1}y_j)$$

$$= \quad x_1(x_1'x_1a_1a_1'a_2a_2')a_2a_3'(\prod_{i=3}^{j-1}(a_{2i-2}a_{2i-1}'))(a_{2j-1}y_j)$$

(since $x_1'x_1, a_1a_1', a_2a_2' \in E(S)$)

$$= \quad x_1a_1a_1'a_2a_3'(\prod_{i=3}^{j-1}(a_{2i-2}a_{2i-1}'))(a_{2j-1}y_j) \qquad \text{(since } x_1, a_2 \text{ are regular)}$$

$$= \quad a_0a_1'a_2a_3'(\prod_{i=3}^{j-1}(a_{2i-2}a_{2i-1}'))(a_{2j-1}y_j) \qquad \text{(by zigzag equations)}$$

$$= \quad (\prod_{i=1}^{j}(a_{2i-2}a_{2i-1}'))(a_{2j-1}y_j).$$

Theorem 6.2.23. Let $\mathcal{V}$ be the class of all $\mathcal{R}$-unipotent semigroups and $\mathcal{C}$ be the class of all left semiregular orthodox semigroups. Then $\mathcal{V}$ is $\mathcal{C}$-closed.

Proof. Let U and S be a $\mathcal{R}$-unipotent semigroup and a left semiregular orthodox semigroup respectively with U a subsemigroup of S. Take any $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.
Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)} \\
&= x_1a_1a_1'a_1y_1 && \text{(since } a_1 \text{ is regular)} \\
&= a_0a_1'(a_1y_1) && \text{(since } a_1 \text{ is regular)} \\
&= (a_0a_1')(a_2a_3')(a_3y_2) && \text{(by lemma 2.1)} \\
&= \prod_{i=1}^{2}(a_{2i-2}a_{2i-1}')(a_3y_2)
\end{aligned}$$

$$\vdots$$

$$= \prod_{i=1}^{m-1}(a_{2i-2}a'_{2i-1})(a_{2m-3}y_{m-1})$$

$$= \prod_{i=0}^{m}(a_{2i-2}a'_{2i-1})(a_{2m-1}y_m) \qquad \text{(by lemma 2.1)}$$

$$= \prod_{i=0}^{m}(a_{2i-2}a'_{2i-1})(a_{2m}) \in U \qquad \text{(by zigzag equations)}$$

$\Rightarrow\ d \in U$. Hence $Dom(U, S) = U$. □

Dually, we may state the following:

Theorem 6.2.24. Let $\mathcal{V}$ be a class of all $\mathcal{L}$-unipotent semigroups and $\mathcal{C}$ be a class of all right semiregular orthodox semigroups with $\mathcal{V}$ a subclass of $\mathcal{C}$. Then $\mathcal{V}$ is closed in $\mathcal{C}$. □

Corollary 6.2.25. Let $\mathcal{V}$ be a class of all $\mathcal{R}$-unipotent semigroups and $\mathcal{C}$ be a class of all quasi-inverse semigroups with $\mathcal{V}$ a subclass of $\mathcal{C}$. Then $\mathcal{V}$ is closed in $\mathcal{C}$.

Since the class of WL[WR]- regular orthodox semigroups is contained in the class of quasi-inverse semigroups, it is natural to ask whether the class of all WL[WR]-regular orthodox semigroups is closed in the class of all quasi-inverse semigroups. We have not been able to answer this question. However, in the following, we prove that the class of all WL[WR]-regular orthodox semigroups is closed and, thus, generalize [2, Corollary 2.5].

Theorem 6.2.26. The class of all WL-regular orthodox semigroups is closed.

Proof. Let U and S be WL-regular orthodox semigroups with U a subsemigroup of S. Take $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.
Now

$$d = a_0 y_1$$

$$
\begin{aligned}
&= x_1a_1y_1 && \text{(by zigzag equations)}\\
&= x_1a_1a_1'a_1y_1 && \text{(since } a_1 \text{ is regular)}\\
&= x_1a_1a_1'a_2y_2 && \text{(by zigzag equations)}\\
&= x_1(x_1'x_1a_1a_1'a_2a_2')a_2y_2 && \text{(since } x_1 \text{ and } a_2 \text{ are regular)}\\
&= x_1(x_1'x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_1)a_2y_2 && \text{(since } x_1'x_1, a_1a_1', a_2a_2' \in E(S)\\
& && \text{and } E(S) \text{ is WL-regular band)}\\
&= x_1x_1'x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_2a_3y_2 && \text{(by zigzag equations)}\\
&= x_1x_1'x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_2a_3a_3'a_3y_2 && \text{(since } a_3 \text{ is regular)}\\
&= x_1x_1'x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_1a_2a_3'a_3y_2 && \text{(by zigzag equations)}\\
&= x_1x_1'x_1a_1a_1'a_2a_2'a_2a_3'a_3y_2 && \text{(since } x_1'x_1, a_1a_1', a_2a_2' \in E(S))\\
&= x_1a_1a_1'a_2a_3'a_3y_2 && \text{(since } x_1, a_2 \text{ is regular)}\\
&= a_0a_1'a_2a_3'(a_3y_2) && \text{(by zigzag equations)}\\
&\;\vdots\\
&= a_0a_1'a_2a_3'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'a_2a_3'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_1a_2a_3'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
& && \text{(since } x_1'x_1, a_1a_1', a_2a_2' \in E(S))
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_2a_3a_3'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&\ \vdots\\
&= x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_2a_3a_3'\cdots x_{m-2}a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= z\cdots x_{m-2}a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(where } z = x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_2a_3a_3')\\
&= z\cdots(x_{m-1}a_{2m-3}a_{2m-3}')(a_{2m-2}y_m) && \text{(by zigzag equations)}\\
&= z\cdots(x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}')(a_{2m-2}y_m) && \text{(since } a_{2m-2} \text{ is regular)}\\
&= z\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-2}'x_{m-1}'x_{m-1}a_{2m-2}y_m\\
& && \text{(since } x_{m-1}'x_{m-1}, a_{2m-3}a_{2m-3}', a_{2m-2}a_{2m-2}' \in E(S))\\
& && \text{and } E(S) \text{ is WL-regular band)}\\
&= z\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-2}'x_{m-1}'x_ma_{2m-1}y_m\\
& && \text{(by zigzag equations)}\\
&= z\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-2}'x_{m-1}'x_ma_{2m-1}a_{2m-1}'a_{2m-1}y_m\\
& && \text{(since } a_{2m-1} \text{ is regular)}\\
&= z\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-2}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-1}'a_{2m}\\
& && \text{(by zigzag equations)}\\
&= z\cdots x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}(a_{2m-2}'a_{2m-2}a_{2m-2}'a_{2m-1}'a_{2m}\\
& && \text{(since } x_{m-1}'x_{m-1}, a_{2m-3}a_{2m-3}', a_{2m-2}a_{2m-2}' \in E(S))\\
&= z\cdots x_{m-2}a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} && \text{(by zigzag equations)}\\
&\ \vdots\\
&= x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_2a_3a_3'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}\\
& && \text{(as } z = x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_2a_3a_3')
\end{aligned}
$$

$$= x_1a_1a_1'x_1'x_1a_2a_2'x_1'x_1a_2a_3' \cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}$$

(by zigzag equations as $x_2a_3 = x_1a_2$)

$$= x_1a_1a_1'a_2a_2'a_2a_3' \cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} \quad \text{(since } x_1'x_1, a_1a_1', a_2a_2' \in E(S))$$

$$= a_0a_1'a_2a_3' \cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} \in U \quad \text{(by zigzag equations)}$$

$\Rightarrow d \in U$. Hence $Dom(U,S) = U$. □

Dually, we may prove the following:

Theorem 6.2.27. The class of all WR-regular orthodox semigroups is closed. □

In [2], authors have shown that the class of all left[right] quasinormal orthodox semigroups is closed. Now, we generalize this result and show that the class of all left[right] seminormal orthodox semigroups is closed and, as a corollary, we deduce that the class of all left[right] seminormal bands is closed and, thus, generalize Theorems 2.2 and 2.3 of [4].

Theorem 6.2.28. The class of all left seminormal orthodox semigroups is closed.

Proof. Let U and S be left seminormal orthodox semigroups with U a subsemigroup of S. Take $d \in Dom(U,S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.
Now

$$\begin{aligned} d &= a_0y_1 \\ &= x_1a_1y_1 && \text{(by zigzag equations)} \\ &= x_1a_1a_1'a_1y_1 && \text{(since } a_1 \text{ is regular)} \\ &= x_1a_1a_1'a_2y_2 && \text{(by zigzag equations)} \end{aligned}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'a_2y_2 \qquad \text{(since } x_1 \text{ and } a_2 \text{ are regular)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2a_2'a_2y_2 \qquad \text{(since } x_1'x_1, a_1a_1', a_2a_2' \in E(S)$$
$$\text{and } E(S) \text{ is left seminormal band)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2y_2 \qquad \text{(since } a_2 \text{ is regular)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_2a_3y_2 \qquad \text{(by zigzag equations)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_2a_3a_3'a_3y_2 \qquad \text{(since } a_3 \text{ is regular)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2a_3'a_3y_2 \qquad \text{(by zigzag equations)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2a_2'a_2a_3'a_3y_2 \quad \text{(since } a_2 \text{ is regular)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'a_2a_3'a_3y_2 \qquad \text{(since } x_1'x_1, a_1a_1', a_2a_2' \in E(S))$$
$$\text{and } E(S) \text{ is left seminormal band)}$$

$$= x_1a_1a_1'a_2a_3'(a_3y_2) \qquad \text{(since } x_1 \text{ and } a_2 \text{ are regular)}$$

$$= a_0a_1'a_2a_3'(a_3y_2) \qquad \text{(by zigzag equations)}$$

$$\vdots$$

$$= a_0a_1'a_2a_3'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})$$

$$= x_1a_1a_1'a_2a_3'\cdots a_{2m-4}a_{2m-3}'a_{2m-3}y_{m-1} \qquad \text{(by zigzag equations)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'a_2a_3'\cdots a_{2m-4}a_{2m-3}'a_{2m-3}y_{m-1} \quad \text{(since } x_1 \text{ and } a_2 \text{ are regular)}$$

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2a_2'a_2a_3'\cdots a_{2m-4}a_{2m-3}'a_{2m-3}y_{m-1}$$

(since $x_1'x_1, a_1a_1', a_2a_2' \in E(S)$)
and $E(S)$ is left seminormal band)

$$= x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2a_2'a_2a_3'\cdots x_{m-2}a_{2m-4}a_{2m-3}'a_{2m-3}y_{m-1}$$

$$= w\cdots x_{m-2}a_{2m-4}a_{2m-3}'a_{2m-3}y_{m-1} \quad \text{(where } w = x_1x_1'x_1a_1a_1'a_2a_2'x_1'x_1a_2a_2'a_2a_3')$$

$$= w\cdots x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}y_m \quad \text{(by zigzag equations)}$$

$$= w\cdots x_{m-1}(x_{m-1}'x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}')a_{2m-2}y_m$$

(since x_{m-1} and a_{2m-2} are regular)

$$= w\cdots x_{m-1}(x_{m-1}'x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-2}')a_{2m-2}y_m$$

(since $x_{m-1}'x_{m-1}, a_{2m-3}a_{2m-3}', a_{2m-2}a_{2m-2}' \in E(S)$
and $E(S)$ is left seminormal band)

$$= w\cdots x_{m-1}(x_{m-1}'x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}'x_{m-1}'x_{m-1}a_{2m-2}y_m$$

$$= w\cdots x_{m-1}x_{m-1}'x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}'x_{m-1}'x_ma_{2m-1}y_m$$

(by zigzag equations)

$$= w\cdots x_{m-1}x_{m-1}'x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}'x_{m-1}'x_ma_{2m-1}a_{2m-1}'a_{2m-1}y_m$$

(since a_{2m-1} is regular)

$$= w\cdots x_{m-1}x_{m-1}'x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-1}'a_{2m}$$

(by zigzag equations)

$$= w\cdots x_{m-1}x_{m-1}'x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-2}'a_{2m-2}a_{2m-1}'a_{2m}$$

(since $x_{m-1}'x_{m-1}, a_{2m-3}a_{2m-3}', a_{2m-2}a_{2m-2}' \in E(S)$)

$$= \quad w \cdots x_{m-1}a_{2m-3}a'_{2m-3}a_{2m-2}a'_{2m-1}a_{2m} \qquad \text{(since } x_{m-1} \text{ and } a_{2m-2} \text{ are regular)}$$

$$= \quad w \cdots x_{m-2}a_{2m-4}a'_{2m-3}a_{2m-2}a'_{2m-1}a_{2m} \qquad \text{(by zigzag equations)}$$

$$\vdots$$

$$= \quad w \cdots a_{2m-4}a'_{2m-3}a_{2m-2}a'_{2m-1}a_{2m}$$

$$= \quad x_1x'_1x_1a_1a'_1a_2a'_2x'_1x_1a_2a'_2a_2a'_3 \cdots a_{2m-4}a'_{2m-3}a_{2m-2}a'_{2m-1}a_{2m}$$
$$\text{(as } w = x_1x'_1x_1a_1a'_1a_2a'_2x'_1x_1a_2a'_2a_2a'_3\text{)}$$

$$= \quad x_1x'_1x_1a_1a'_1a_2a'_2a_2a'_3 \cdots a_{2m-4}a'_{2m-3}a_{2m-2}a'_{2m-1}a_{2m}$$
$$\text{(since } x'_1x_1, a_1a'_1, a_2a'_2 \in E(S) \text{ and } E(S) \text{ is a left seminormal band)}$$

$$= \quad x_1a_1a'_1a_2a'_3 \cdots a_{2m-4}a'_{2m-3}a_{2m-2}a'_{2m-1}a_{2m} \qquad \text{(since } x_1 \text{ and } a_2 \text{ are regular)}$$

$$= \quad a_0a'_1a_2a'_3 \cdots a_{2m-4}a'_{2m-3}a_{2m-2}a'_{2m-1}a_{2m} \in U \qquad \text{(by zigzag equations)}$$

$\Rightarrow$ $d \in U$. Hence $Dom(U, S) = U$. □

Dually, we may prove the following:

Theorem 6.2.29. The class of all right seminormal orthodox semigroups is closed. □

Corollary 6.2.30. The class of all left[right] seminormal bands is closed.

§ 6.3. EMBEDDING OF SEMIGROUP AMALGAMS OF A CLASS OF BANDS INTO THE CLASS OF SEMIGROUPS SATISFYING SOME HOMOTYPICAL IDENTITIES

In this section, we first prove that the class of all normal bands is closed within the class of all medial semigroups which, generalizes, [80, Theorem 4.1], where he has shown that the class of all normal bands is closed within the class of all normal bands.

We, then, prove that the class of all left[right] seminormal bands is closed within the class of all semigroups satisfying the identity $axy = axyay[yxa = yayxa]$. From this , we have deduced, as a corollary, that the class of all left[right] seminormal bands is closed and thus, generalize [4, Theorem 2.4].

Theorem 6.3.1. Let $\mathcal{B}$ be the class of all normal bands and $\mathcal{C}$ be the class of all medial semigroups. Then $\mathcal{B}$ is $\mathcal{C}$-closed.

Proof. Let U and S be a normal band and medial semigroup respectively with U a subsemigroup of S. Let us take $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.
Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations as } a_0 = x_1a_1) \\
&= x_1a_1a_2y_2 && \text{(by zigzag equations as } a_1 \text{ is an idempotent and } a_1y_1 = a_2y_2) \\
&= x_1a_2a_1y_2 && \text{(since } S \text{ is medial)} \\
&= x_2a_3a_1y_2 && \text{(by zigzag equations as } x_1a_2 = x_2a_3) \\
&= x_1a_2a_3a_1y_2 && \text{(by zigzag equations as } a_3 \text{ is an idempotent and } x_2a_3 = x_1a_2) \\
&= x_1a_1a_2a_3y_2 && \text{(since } S \text{ is medial)} \\
&= a_0a_2(a_3y_2) && \text{(by zigzag equations)} \\
&= (\prod_{i=0}^{1} a_{2i})(a_3y_2)
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&= (\prod_{i=0}^{m-2} a_{2i})(a_{2m-3}y_{m-1}) \\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}y_m \qquad \text{(by zigzag equations as } a_{2m-3}y_{m-1} = a_{2m-2}y_m\text{)} \\
&= x_1a_2a_1a_4\cdots a_{2m-4}a_{2m-2}y_m \qquad \text{(since } S \text{ is medial)} \\
&= x_2a_3a_1a_4\cdots a_{2m-4}a_{2m-2}y_m \qquad \text{(by zigzag equations as } x_1a_2 = x_2a_3\text{)} \\
&\vdots \\
&= x_{m-1}a_{2m-3}a_{2m-5}\cdots a_1a_4\cdots a_3a_1a_{2m-2}y_m \\
&= x_{m-1}a_{2m-2}a_{2m-3}a_{2m-5}\cdots a_1a_4\cdots a_3a_1y_m \qquad \text{(since } S \text{ is medial)} \\
&= x_ma_{2m-1}a_{2m-3}a_{2m-5}\cdots a_1a_4\cdots a_3a_1y_m \\
&\qquad\qquad \text{(by zigzag equations as } x_{m-1}a_{2m-2} = x_ma_{2m-1}\text{)} \\
&= x_{m-1}a_{2m-2}a_{2m-3}a_{2m-5}\cdots a_1a_4\cdots a_3a_1a_{2m-1}y_m \\
&\vdots \\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m \\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} \qquad \text{(by zigzag equations as } a_{2m-1}y_m = a_{2m}\text{)} \\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} \qquad \text{(by zigzag equations)} \\
&= (\prod_{i=0}^{m} a_{2i}) \in U. \\
&\Rightarrow d \in U.
\end{aligned}
$$

Hence, $Dom(U,S) = U$. $\square$

Theorem 6.3.2. Let $\mathcal{B}$ be the class of all left seminormal bands and $\mathcal{C}$ be the class of all semigroups satisfying the identity $axy = axyay$. Then $\mathcal{B}$ is $\mathcal{C}$-closed.

Proof. Let U and S be a left seminormal band and a semigroup satisfying the identity $axy = axyay$ respectively with U a subsemigroup of S. Let us take $d \in Dom(U, S) \setminus U$. Then, by Result 1.4.5, we may let (1.4.1) be a zigzag in S over U with value d of minimal length m.

Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations as } a_0 = x_1a_1) \\
&= (x_1a_1a_2)y_2 && \text{(by zigzag equations as } a_1 \text{ is an idempotent and } a_1y_1 = a_2y_2) \\
&= (x_1a_1a_2x_1a_2)y_2 && \text{(by definition of } S) \\
&= x_1a_1a_2x_2a_3y_2 && \text{(by zigzag equations as } x_1a_2 = x_2a_3) \\
&= x_1a_1a_2x_2a_3a_3y_2 && \text{(as } a_3 \text{ is an idempotent)} \\
&= (x_1a_1a_2x_1a_2)a_3y_2 && \text{(by zigzag equations as } x_1a_2 = x_2a_3) \\
&= (x_1a_1a_2)a_3y_2 && \text{(by definition of } S) \\
&= a_0a_2(a_3y_2) && \text{(by zigzag equations)} \\
&= \Big(\prod_{i=0}^{1} a_{2i}\Big)(a_3y_2) \\
&\;\;\vdots \\
&= \Big(\prod_{i=0}^{m-2} a_{2i}\Big)(a_{2m-3}y_{m-1})
\end{aligned}$$

$$\begin{aligned}
&= (x_1a_1a_2)a_4\cdots a_{2m-4}(a_{2m-2}y_m) \text{(by zigzag equations as } a_{2m-3}y_{m-1} = a_{2m-2}y_m)\\
&= (x_1a_1a_2x_1a_2)a_4\cdots a_{2m-4}(a_{2m-2}y_m) \qquad \text{(by definition of } S)\\
&= (x_1a_1a_2x_2a_3)a_4\cdots a_{2m-4}(a_{2m-2}y_m) \qquad \text{(by zigzag equations as } x_1a_2 = x_2a_3)\\
&= x_1a_1a_2(x_2a_3a_4)\cdots a_{2m-4}(a_{2m-2}y_m)\\
&= x_1a_1a_2(x_2a_3a_4x_2a_4)\cdots a_{2m-4}(a_{2m-2}y_m) \quad \text{(by definition of } S)\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots a_{2m-4}(a_{2m-2}y_m) \quad \text{(by zigzag equations as } x_2a_4 = x_3a_5)\\
&\vdots\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots x_{m-2}a_{2m-4}(a_{2m-2}y_m)\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2})y_m\\
&\qquad\qquad \text{(by zigzag equations as } x_{m-2}a_{2m-4} = x_{m-1}a_{2m-3})\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2}x_{m-1}a_{2m-2})y_m \qquad \text{(by definition of } S)\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots x_{m-1}a_{2m-3}a_{2m-2}x_ma_{2m-1}y_m\\
&\qquad\qquad \text{(by zigzag equations as } x_{m-1}a_{2m-2} = x_ma_{2m-1})\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2}x_{m-1}a_{2m-2})a_{2m-1}y_m\\
&\qquad \text{(by zigzag equations as } a_{2m-1} \text{ is an idempotent and } x_{m-1}a_{2m-2} = x_ma_{2m-1})\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots(x_{m-1}a_{2m-3}a_{2m-2})a_{2m-1}y_m \qquad \text{(by definition of } S)\\
&= x_1a_1a_2x_2a_3a_4x_3a_5\cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m-1}y_m\\
&\qquad\qquad \text{(by zigzag equations as } x_{m-2}a_{2m-4} = x_{m-1}a_{2m-3})
\end{aligned}$$

$$\vdots$$

$$= (x_1a_1a_2x_1a_2)a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m$$

$$= (x_1a_1a_2)a_4\cdots a_{2m-4}a_{2m-2}a_{2m-1}y_m \qquad \text{(by definition of } S\text{)}$$

$$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} \qquad \text{(by zigzag equations)}$$

$$= (\prod_{i=0}^{m} a_{2i}) \in U.$$

$$\Rightarrow d \in U.$$

Hence, $Dom(U, S) = U$. □

Dually we may prove the following :

Theorem 6.3.3. Let $\mathcal{B}$ be the class of all right seminormal bands and $\mathcal{C}$ be the class of all semigroups satisfying the identity $yxa = yayxa$. Then $\mathcal{B}$ is $\mathcal{C}$-closed. □

Corollary 6.3.4. The class of all left[right] seminormal bands is closed.

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